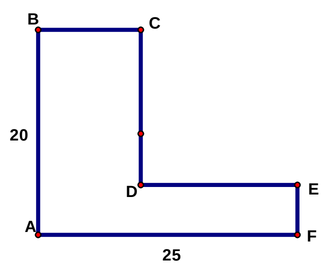
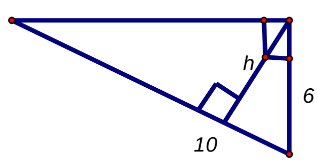
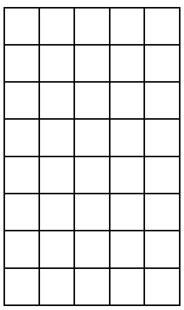
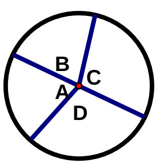
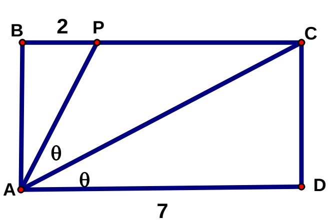
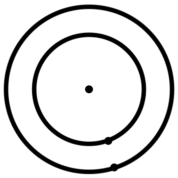
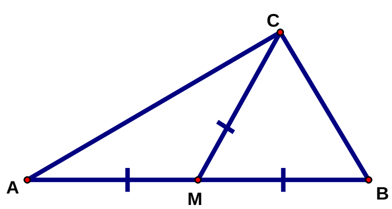
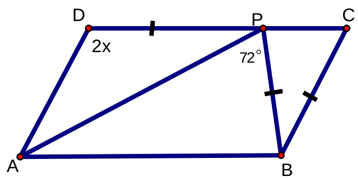
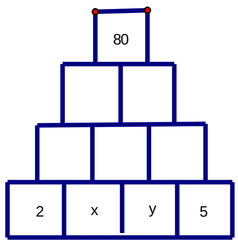
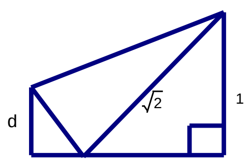
#	L	QUESTION	REF
1	1	Factorise $3^{10} - 1$.	L1.2
2	2	Which of the following numbers is the smallest? 0,2013 ; 0,201 ; 0,0201; 0,02013; 0,02	
3	1	Themba and James sell cakes at school socials. At the first social, they sold 50 cakes, and at the second social they sold 58 cakes. What was the percentage increase in their sales?	
4	3	A group of children see a herd of cattle in the veld. They count the total number of legs and the total number of ears of the cattle. The difference between these two numbers is 92. How many animals are there in the herd?	
5	3	The desks in a classroom are lined up in straight rows. Vusani's desk is in the third row from the front and the fourth row from the back of the classroom. His desk is also the fourth from the left and the sixth from the right. How many desks are there in the classroom?	
6	3	Pieter and Jacob share a packet of sweets in the ratio 7 : 5. Pieter gets 14 sweets more than Jacob. How many sweets were there in the packet?	
7	2	 *ABCDEF* is the figure below (not drawn to scale and angles are right angles). What is the perimeter of the figure?	
8	2	A fraction that lies between $\frac{4}{7}$ & $\frac{5}{8}$ is: $\frac{19}{56};\frac{32}{56};\frac{33}{56};\frac{35}{56};\frac{37}{56}$	
9	4	 Calculate *h*.	
10	4	Mary has three types of toys: teddy bears, cars and jets. All her toys except 21 are jets. All her toys except 23 are teddy bears. All her toys except 26 are cars. How many jets does she have?	

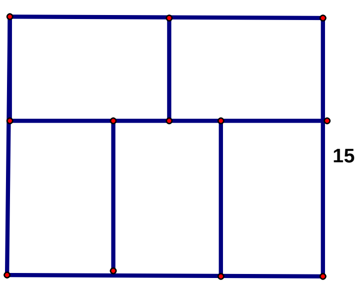
11	4	1287a45b is an 8-digit number, where a and b are not zero. The number is divisible by 18. What is the maximum possible difference between a and b?	L1.5
12	4	A bead worker is threading beads onto a straight wire; he has four green beads and two red beads and will use them all. How many different arrangements he can make?	L5.1 - 5.3
14	3	If January 1st 1985 was a Tuesday, how many Tuesdays were there in 1985?	L8.5
15	1	How many zeros are there in 3001^2, when written in full?	
16	1	The three digit number 7d2 is divisible by 11. Determine d.	L1.5
17	3	Calculate (2 + 4 + 6…+ 50) - (1 + 3 + 5 +…+ 49)	
18	3	As n gets larger and larger, what does the value of $\frac{n+2}{2n+1}$ gets closer and closer to?	
19	2	 A bathroom floor is covered with square tiles. The floor is 5 tiles wide and 8 tiles long. If one of the floor tiles is chosen at random, what is the probability that it is at the edge of the floor?	L5.7
20	3	A circle is divided into four sectors. $\angle A=\frac{2}{3}\angle C$, $\angle D=2\angle B.$ Angles B and C are supplementary. Calculate the size of $\angle C$ 	
21	2	ABCD is a rectangle. BP = 2, AD = 7, $\angle PAC=\angle CAD$. Calculate the length of AP. 	L15
22	3	The decimal form of $3\div 7$ is the recurring decimal 0,428571428571…. The digit in the 2013th place is 4; 2; 8; 5; 7	

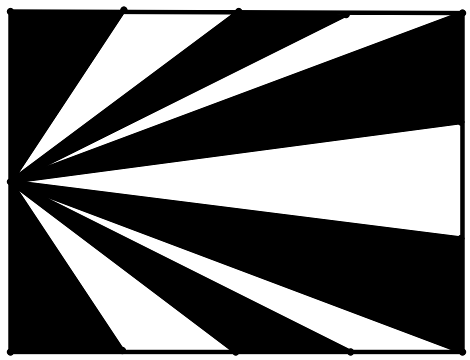
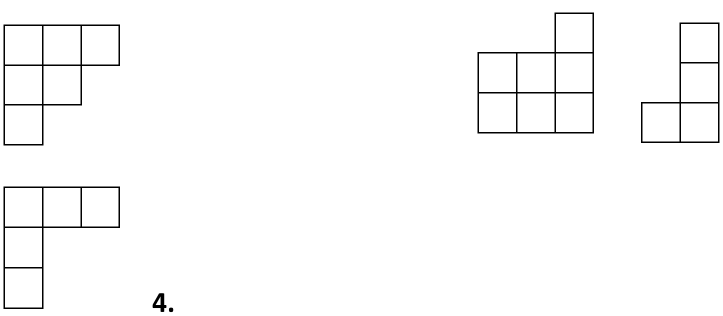
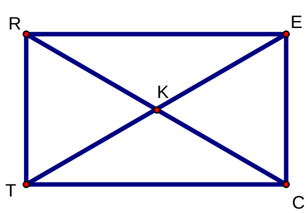
23	4	 The diagram shows two concentric circles. If the circumference of one exceeds the other by 6cm by how much does the radius exceed the other radius?	L15
24	3	Calculate : $(1-\frac{1}{3})(1-\frac{1}{4})(1-\frac{1}{5})...(1-\frac{1}{100})$	
25	3	A petrol tank weighs 34kg when empty and 58kg when full. Calculate its weight when it is two-thirds full?	
26	3	A set of 12 numbers has average 18, but the smallest and largest have average 28. What is the average of the others?	
27	5	Four teams play in a knockout tournament (which means that two pairs compete, and the two winners then play each other). Team A beat team D, and team B beat team A. Who beat team C?	
28	3	In the sequence 5, 11, 17… how many terms are smaller than 1000?	, L8.5
29	3	M is the midpoint of AB and joined to the third vertex of ΔABC, with MC=AM=MB. Calculate x + y. 	
30	4	It has been observed that in a herd of gazelle, there is always at least one male for every 5 females. If m is the number of males and the number f of females is a multiple of 5, which is true? $m \geq 5f$; $5m \geq f$; $m \leq 5f$; $5m \leq f$; $m+f \geq 5$	
31	5	The positive integers are written in a long sequence 12345678910111213….When the sequence contains 100 digits, how many of those are 1's?	
32	3	How many factors does 11x13x17x19 have?	L1.4
33	3	A certain value increases by a fixed amount each year. If the percentage increase during the first year was 10%, what was the percentage increase during the third year?	

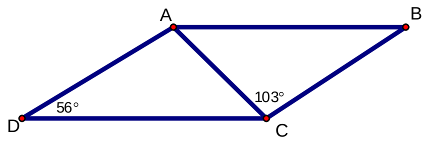
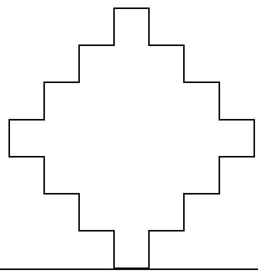
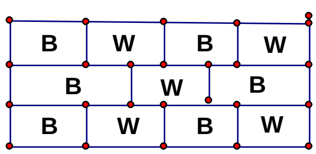
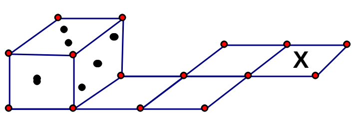
<table>
<tr><td>34</td><td>3</td><td>ABCD is a parallelogram. BP = DP = BC.


Calculate x.</td><td></td></tr>
<tr><td>35</td><td>3</td><td>I have five books, one of each colour. In how many ways can I place them in a row?</td><td>L4.2</td></tr>
<tr><td>36</td><td>3</td><td>

The number in each box is the product of the numbers in the two boxes below it.
Calculate xy.</td><td></td></tr>
<tr><td>37</td><td>4</td><td>

A rectangular sheet of paper with sides $\sqrt{2}$ and 1 has been folded as shown, so that one corner meets the opposite long edge. Calculate d.</td><td>L15</td></tr>
<tr><td>38</td><td>3</td><td>Five identical rectangles are placed to form a new rectangle. The width of the new rectangle is 15cm. The area of the big rectangle (in square cm) is:

</td><td></td></tr>
</table>

39	4	 The long sides of a rectangle are divided into four equal parts and the short sides into three equal parts. One point on a short side is joined to all the others. What fraction of the area of the large rectangle is the area of the shaded region?	L15
40	2	Which two of the shapes can be joined to form a rectangle? (No gaps or overlapping allowed). **1.** **2.** **3.**  **4.**	
41	2	 The diagonals of rectangle RECT intersect at K. If RE = 4 and EC = 3, determine RK.	L15
42	3	$2013^2 - 1$ equals … 2012 x 2014; 2012 x 2013; 2013 x 2014; 2012 x 2012; 2000 x 2013	
43	3	The largest number in the following list is $\frac{3}{2^3}$; $\frac{1}{4}$; $(0.2)^3$; $\frac{7}{16}$; 0,625	
44	3	Barbie's bag is half full with 30 tennis balls in it. If she takes half of the balls from Ken's bag, then her bag is two-thirds full. How many tennis balls were there originally in Ken's bag?	

45	2	 If ABCD is a parallelogram, determine angle $\angle ACD$.	L15
46	3	A coin and a die are thrown simultaneously. What is the probability of getting a tail and a 6?	L5.7
47	4	 All 28 sides of the polygon are equal in length and adjacent sides are perpendicular to each other. If the perimeter of the polygon is 56cm, calculate the area of the polygon.	
48	4	A flag is made up of three strips of equal width. Each strip is divided into equal parts with alternating black and white colors, as shown. What fraction of the flag is black in colour? 	
49	5	 On a die the numbers on opposite faces add up to 7. The die in the diagram is rolled edge over edge along the path until it rests on the square labelled X. In that position, what is the number on top?	
50	5	Children at a school fun day had to guess the number of marbles in a jar. Prizes were awarded on how close the guesses were. The first prize went to Emily who guessed 125 marbles, second prize to Cassandra who guessed 140, third prize to Sigrid who guessed 142, and fourth prize to Rina who guessed 121. How many marbles were in the jar?	

LESSON 1

PRIME NUMBERS AND FACTORISATION

1.1 FACTORS:

The natural numbers are the numbers 1, 2, 3, 4, ….

The integers are the naturals numbers together with 0 and the negative integers. That is the integers are …-3, -2, -1, 0, 1, 2, 3,….

This lesson deals mainly with natural numbers and, sometimes, integers. The study of integers is called **number theory.**

The sum and product of integers (or natural numbers, for that matter), is an integer (natural number).

You will agree that 4 is a **factor** of 8, 6 is a factor of 18, 7 is a factor of 35, and so on. Is -7 a factor of 35? So, what do we really mean by the expression "**a is a factor of b**"?.

One way of answering this is:

1.1.1 Let a and b be integers, where $a \neq 0.$ then a is a factor of b if $\frac{b}{a}$ is an integer.

So the answer to the last question is yes, since -5 is an integer. We could say that a factor is a positive integer (which is exactly the same as a natural number), if we so wish.

If we give a name to $\frac{b}{a}$, call it n for example, we can say $\frac{b}{a} = n$, where n is an integer.

Or we can take it further: **a is a factor of b if b = an where n is an integer.** While all these can serve as definitions for the statement " a is a factor of b", the last one is the one that is used most.

For example, the positive factors of 15 are 1, 3, 5 and 15. However the answer to

"Find *all* the factors of 15. " is : 1, -1, 3. -3, 5, -5, 15, -15.

1. Instead of using the cumbersome "a is a factor of b" we often write a|b to mean the same thing. Hence 4|8, 6|18, 7|35 etc.
2. Factors have some very nice and useful properties. Some of them are:

2.1 1|b for any integer b

2.2 b|b for any integer b

2.3 If a|b and b|c, then a|c

2.4 If $a|b$ and $a|c$ then $a|b + c$ and $a|b - c$

2.5 Always, $a|ab$.

Learn these properties by reading them out in sentences and understanding them.

1.2 <u>EXAMPLES</u>

1. Write down all the positive factors of 12. Then write <u>all</u> the factors.
   <u>Solutions</u>
   12 = 1.12 = 2.6 = 3.4. The positive factors are 1, 12, 2, 6, 3, 4.
   The factors are 1, -1, 12, -12, 2, -2, 6, -6, 3, -3, 4, -4.

2. Write down all the positive factors of 60.
   <u>Solution</u>:
   60 =1.60 = 2.30 = 3.20 = 4.15 =5.12 = 6.10
   The factors are 1, 60, 2, 30, 3, 20, 4, 15, 5, 12, 6, 10. There are 12 of them.

3. Determine all integers n that have the property that $\frac{35}{n-4}$ is an integer.

   <u>Solution</u>: **Notice that the statement "** $\frac{35}{n-4}$ is an integer" is exactly the same as the statement "n – 4 is a factor of 35"! So n – 4 is one of the numbers 1, -1, 5, -5, 7, -7, 35, -35, from which we deduce that n is one of the numbers 5, 3, 9, -1, 11, -3, 39, -31. There are eight solutions for n.

1.3 <u>PRIME NUMBERS</u>

1.3.1 A **prime number** is a number **greater** than 1, that has <u>exactly</u> two factors, namely, 1 and itself. Or, put in another way, a number that cannot be factorised any further.

One of the most important properties of prime numbers is the following:

1.3.2 Any natural number can be written as a product of prime numbers, and there is only one such factorisation, if we insist that the primes that occur in the factorisation are written in increasing order.

1.3.3 A number which is not a prime number is a **composite number.** Study the examples below carefully. They inform you on how to factorise numbers into products of prime *powers*.

$120 = 12.10 = 3.4.10 = 3.2.2.2.5 = 2^3.3.5$

$3600 = 36.100 = 4.9.10.10 = 2.2.3.3.2.5.2.5 = 2^4 3^2 5^2$.

$3600 = 60.60 = 6.10.6.10 = 2.3.2.5.2.3.2.5 = 2^4 3^2 5^2$.

$1000 = 10^3 = 2^3.5^3$

2013 = 3. 671 = 3.11.61

2014 = 2.1007 = 2. 19. 53

2015 = 5.403 = 5.13.31

1.3.4 Every natural number n can be written in the form $2^m d$ where d is the largest odd factor of n

Proof: Let n be a natural number. Factorise as a product of prime numbers. Then n = 2^md (m may be zero, if n is odd for example) where d is a product of prime numbers greater or equal to 3, and hence odd. Clearly d is the largest odd factor of n.

1.4 NUMBER OF DISTINCT (POSITIVE) FACTORS OF A NUMBER

Let n be the number under investigation. How many distinct factors does n have?

Firstly suppose n is prime number. Then it has two factors, namely 1 and n. Next suppose n is a power of a prime number. For example, suppose

n = 3^4 =81. Then its factors are 1, 81, 3, 27, 9. They are all, of course powers of 3, that is,

1, 3, 3^2, 3^3, 3^4 which simplify to 1, 3, 9, 27 and 81. So 3^4 has 5 factors. If 3 was replaced by another prime number, say 5, we will again obtain 5 factors, namely, 1, 5, 5^2,5^3 and 5^4. It should be now clear that

1.4.1 If p is any prime number then p^m has m + 1 factors, namely, 1, p, p^2, p^3, p^4,…..p^m.

Now, any natural number can be written (uniquely) as a product of powers of primes, as we have already seen.

Example: For example, 400 = 4 x 100 = 16 x 25 = $2^4.5^2$. How does one count the number of its factors? One way is to go through the factors in pairs:

1, 400, 2, 200, 4, 100, 5, 80, 8, 50, 10, 40, 16, 25, 20
giving 15 factors altogether.

Well, fix a factor of 5^2 say, 5. There are five factors of 400 associated with 5, namely,

1.5, 2.5, 2^2.5, 2^3.5, 2^4.5

This can be done for **each factor of the three factors of 5^2, namely 1, 5 and 25,**. They are

1	2	2^2	2^3	2^4
1.5	2.5	$2^2.5$	$2^3.5$	$2^4.5$
1.5^2	2.5^2	$2^2.5^2$	$2^3.5^2$	$2^4.5^2$

So again we come to 15 factors. But observe that 15 arises from 15 = 3.5. More precisely, **for each of the five factors** $1, 2, 2^2, 2^3$ & 2^4 **of** 2^4 **three new factors may be obtained obtained, by multiplying each of these five with the three factors of 5^2. Summarising: the number of factors of** $2^4.5^2$ is (4+1)(2+1) = 5.3 = 15.

This observation allows one to calculate the number of prime factors of **any natural number** quickly:

1. Factorise the number into prime powers.
2. If p^n is one of the prime powers in the factorisation, then p^n has n + 1 factors, as we have seen.
3. Increase each of the powers of the primes obtained by 1 and **multiply** all of them.
   The answer you get is the number of factors of the number that was given.

Formally stated, we have:

1.4.2 If $n = p_1^{\ a} p_2^{\ b} p_3^{\ c} .....$ **is the factorisation of *n* into prime powers, then n has (a + 1) (b + 1)(c + 1) …..factors altogether.**

<u>Examples</u>:

1. We saw earlier that both 400 has 15 factors and 60 has 12 factors. Note that

400 = $2^4 5^2$ and (4+1)(2+1) =15

60=6.10 = 2.3.2.5 = 2^2.3.5 and (2 + 1)(1 + 1)(1 + 1) = 12

2. How many factors does 12 800 have?

Answer: 12 800 = $2^9.5^2$ has (9 + 1). (2 + 1) = 30 factors.

1.5 <u>DIVISIBILITY RULES</u>

The following rules provide quick and easy ways of deciding whether one of the numbers from 2, 3, 4, 5 6, 8, 9 or 11 are factors of a given number. We state them now and reserve their proofs for Lesson 3.6

A number is divisible by

- 2 if the last digit is even
- 3 if and only if the sum of its digits is divisible by 3.

- 4 if and only if the last two digits is divisible by 4.
- 5 if the last digit is either 0 or 5.
- 6 if it is divisible by 2 and 3.
- 8 if and only if the last three digits is divisible by 8.
- 9 if and only if the sum of its digits is divisible by 9.
- 11 if and only if the the difference between the “alternate sums” is divisible by 11.

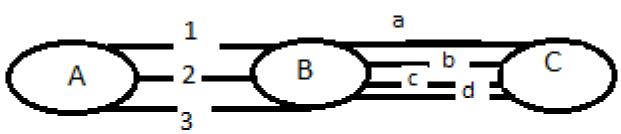
EXAMPLES

- 43527684 is divisible by 2 since 4 (last digit) is even
- 56178951 is divisible by 3 since 5+6+1+7+8+9+5+1 = 42 is divisible by 3
- 34519068 is divisible by 4 since 68 is.
- 9745875 ends in 5 and so is divisible by 5
- 86594384 is divisible by 8 since 384 = 8.48 is.
- 675483264 is divisible by 9 since 6+7+5+4+8+3+2+6+4 = 45 is divisible by 9
- 358296004 is divisible by 11 since (3+8+9+0+4)-(5+2+6+0) = 11 is divisible by 11.

LESSON 4: THE PRODUCT RULE IN COUNTING

4.1 THE PRODUCT RULE IN COUNTING

Suppose A, B and C are three towns. If there are three roads connecting A and B, and four roads connecting B and C, how many different ways are there to get from A to C?



Let us name the roads from A to B by 1, 2 and 3, while the roads from B to C are named a, b, c and d. By 1a, we shall mean the path from A to C, using road 1 followed by road a. If we choose road 1, there are four possible routes from A to C, viz 1a, 1b, 1c and 1d. Likewise, with road 2, we have 2a, 2b, 2c and 2d. And with road three, we have 3a, 3b, 3c and 3d. **Ans: There are 12 ways to get from A to C.**

We could have arrived at 12 using the following argument.

There are* 3 *ways of choosing the first road. For each choice of the first road, there are* 4 *choices for the second road. Hence there are* 3 x 4 = 12 *of choosing both roads together.

We can **generalise**.

> **4.1.1 THE FIRST PRODUCT RULE**: ***Suppose there are m ways of doing thing, and n ways of doing another after the first has been done. Then there are mn ways of doing both things together***.

Examples:

1. How many two lettered "words" can be made using the letters of the word CAT?

Method 1: We can actually make the words.

First letter C: CC, CA, CT

First letter A: AC, AA, AT

First letter T: TC, TA, TT

Answer: 9 words

Method 2 (preferred): The first letter can be chosen in 3 ways. For each choice of the first letter, there are three choices foe the second letter. So there are 3 x 3 = 9 ways of choosing both letters together, that is, of making two lettered words.

2. How many three lettered words can be made using the letters of the word CAT?

Answer: For each choice of the first two letters, there are three for the third. For example, if the first two were chosen as AT, then we can form ATC, ATA and ATT. But there are 9 ways of choosing the first two letters. So there are 9 x 3 or 3 x 3 x 3 = 27 words that can be made.

The above example gives us a method of solving more complex problems.

3. How many 4 lettered words can be made from the word MASTER?

Answer: The first letter can be chosen in 6 ways, the first two in 6 x 6 ways, the first three in 6 x 6 x 6 ways, so the four letters can be 6 x 6 x 6 x 6 = 1296 ways. So there 1296 such words.

We have the following rule:

> **4.1.2 THE SECOND PRODUCT RULE IN COUNTING** : **Suppose we need to fill in r boxes. If the first box can be filled in m_1 ways, the second in m_2 ways, third in m_3 ways, and so on, until the r^th box, which can be filled in m_r ways, then all r boxes can be filled in $m_1 m_2 m_3 \ldots\ldots m_r$ ways.**
>
> | | | | | |
> |---|---|---|---|---|
> | m_1 | m_2 | m_3 | .... | m_r |

4. How many two lettered words can be made from the letters of the word CAT, *if no letter is to be repeated?*

This means that "words": like CC are <u>not allowed.</u>

The words are: CA, CT, AC, AT, TA, TC.
There are 6 such words.

Or, better still:
The first letter can be chosen in 3 ways, but the second in only two ways, <u>since the first cannot be used.</u>

3	*2*

which leads two 3 x 2 = 6.

5. How many 5 lettered words can be made using the letters from A to G only, and no letter is to be repeated? If repetitions are allowed?

<u>Answer</u>: The letters are to be chosen from the seven letters A, B, C, D, E, F and G. The first letter can be chosen in 7 ways, the second in 6 and so on. So there are

7 x 6 x 5 x 4 x 3 = 2520 words

If repetitions are allowed, we have

7	7	7	7	7

<u>Answer</u>: $7^5 = 16807$ words

4.2 <u>ARRANGEMENTS OR PERMUTATIONS</u>

<u>4.2.1 Definition</u>: Let n be a natural number. For the product 1.2.3.4…..n, we write n! That is n! = 1.2.3…..n. (Read n! as "n factorial").

0! = 1

<u>Examples</u>

1. Show that the number of 5 letter "words", with no letters repeated, that can be made from using the letters of the word STRANGE is $\frac{8!}{3!}$

Proof: We need to fill five boxes with no repetitions:

8	7	6	5	4

So the number of words is 8.7.6.5.4. Now

$$8.7.6.5.4 = 4.5.6.7.8 = \frac{1.2.3.4.5.6.7.8}{1.2.3} = \frac{8!}{3!}.$$

4.3 COUNTING ARRANGEMENTS

Generalise (6) above: Show that the number of r lettered "words" that can made from a word having n different letters, repetitions not allowed, is

$$n(n-1)(n-2)...(n-r+1) = \frac{n!}{(n-r)!}$$

Proof: We have to fill r boxes, without repetitions:

n	n-1	n-2	n-3	….	n-r+1

(The second box is n–1, third is n–2, …so the r^{th} box is n–(r –1) = n –r + 1)

Hence the number of such words is

$$n(n-1)(n-2)...(n-r+1)$$
$$= (n-r+1)(n-r+2)....(n-1)n$$
$$= \frac{1.2.3......(n-r)(n-r+1)....(n-1)n}{1.2.3.....(n-r)}$$
$$= \frac{n!}{(n-r)!}$$

Another way of stating the above result, without resorting to "words" is:

> **4.2.2 The number of ways of arranging n objects, taken r at a time is**
>
> $$n(n-1)(n-2)...(n-r+1) = \frac{n!}{(n-r)!}$$

4.4 EXERCISES

1. You are asked to pick the first, second and third athletes in a race in which there are 12 athletes. What is the least number of choices you have to make? If you are given twenty guesses, what is the probability that one of them is correct? (Problem 121)
2. 8 books are to be arranged on a shelf. In how ways can they be arranged?

(Problem 122)

3. 8 books are to be arranged on a shelf so that two particular books are to be at the ends. In how many ways can this be done? (Problem 123)
4. How may 5 digit numbers greater than 60000 are odd? (Problem 124)
5. You wish to create a string of 10 numbers using only 0 and 1. How many such numbers can you create? (Problem 125)

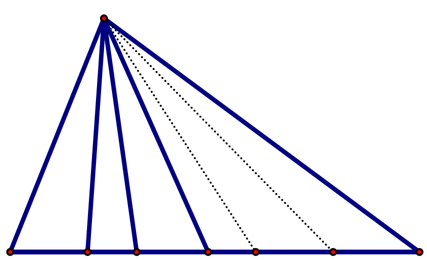
LESSON 5

BINOMIAL COEFFICIENTS

5.1 <u>NUMBERS OF THE FORM</u> $\frac{n(n-1)}{2}$

5.1.1.Reflect on the following problems:

1. Find a formula for $1+2+3+4...(n-1)$ in terms of *n*.
2. Find a formula for $1+2+3+4....+n$
3. Hoe many two-element subsets does a set having n elements have?
4. You have *n* points on a flat surface, no three of which lie on the same straight line. A straight line is drawn through every pair of these points. How many such lines are there?
5. *n* points all lie on the same straight line. A point P that is not on the line is joined to all *n* points. How many triangles are so formed?



6. To win a competition, you have to pick the first two horses in a rece. What is the least number of picks you have to make to be sure of winning, if *n* horses have entered the race?
7. What is the coefficient of x^2 in the expansion of $(1+x)^n$?

<u>Problem 1:</u> Let $x=1+2+3+...+(n-1)$

Write it backwards. The new sum is still x.

$$x=1 \quad + \quad 2 \quad +... +(n-2)+(n-1)$$
$$x=(n-1)+(n-2)+...+ \quad 2 \quad + \quad 1$$

Add both sides:

$$2x = n + n + n + \ldots + n \ldots\ldots\ldots (n-1) \text{ terms}$$
$$= n(n-1)$$
$$x = \frac{n(n-1)}{2}$$

Hence

$$\boxed{1+2+3+4\ldots(n-1) = \frac{n(n-1)}{2}}$$

Problem 2: This is the same as Problem 1, except that *n* – 1 is replaced with *n.*

Answer: $1+2+3+4\ldots.+n = \frac{(n+1)n}{2} = \frac{n(n+1)}{2}$

5.1.2 The sum of the first n natural numbers is

$$1+2+3+4\ldots.+n = \frac{n(n+1)}{2}$$

Problem 3: Let the set be {1, 2, 3, 4, …….*n*}. The two-element subsets are:

{1,2}, {1,3}, {1,4},….{1,n}……………………………there are *n* – 1 of them
{2,3}, {2,4}, {2,5},….{2,n}……………………………. there are *n* – 2 of them
{3,4}, {3,5},{3,6}……{3,n}……………………………. there are *n* – 3 of them
.
.
.
{n,n-1}……………………………………………………. (there is only one)

Note that all the two-element subsets have been counted. And we have

$1+2+3+4\ldots(n-1)$ of them.

So, a set having *n* elements has $1+2+3+4\ldots(n-1) = \frac{n(n-1)}{2}$ **two-element subsets.**

Problem 4: Any two points determine exactly one line, and every such line is associated with exactly one pair of points. So the number of lines is the same as the number of ways two points may be chosen from *n* points. We have seen in problem 3 that this number is $\frac{n(n-1)}{2}$

Problem 5: Three vertices determine a unique triangle; so the number of triangles is equal to the number of ways three points may be chosen. One of them is always P. So we have choose two points from the n points on the line, to create a triangle. The number of ways that this can be done is $\frac{n(n-1)}{2}$

Problem 6 : This is now clear; there are $\frac{n(n-1)}{2}$ ways of picking the two horses.

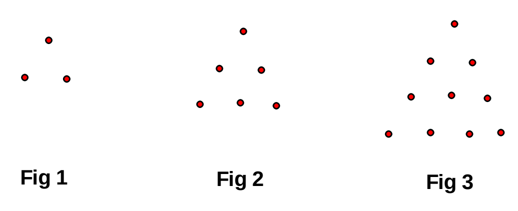
5.1.3 Let *n* be a natural number. That is $n \in \{1,2,3,4,....\}$. A number of the form $\frac{n(n-1)}{2}$ is called a <u>triangular number</u>. It is also written $\binom{n}{2}$ that is, $\binom{n}{2} = \frac{n(n-1)}{2}$

Problem 7: Consider instead the product $(1+x_1)(1+x_2)(1+x_3)....(1+x_n)$. Amongst the terms in its expansion are those which are products of two of the x's, like x_2x_3, x_1x_5, x_4x_3 and so on.How many such terms are there. As many ways as we can choose 2 numbers from 1, 2, 3, …n of course, and thios number is $\binom{n}{2}$ **. Now make all the x's equal to x. That is,** $x = x_1 = x_2 = x_3 ... = x_n$ Then the original product is $(1+x)^n$, all the products of pairs are all equal to x^2, and there are $\binom{n}{2}$ **. of them. So the coefficient of x^2 in the expansion of** $(1+x)^n$ is $\binom{n}{2}$**.**

5.2 <u>Exercises</u>

1. List the first seven triangular numbers. (Problem 149)

2. Consider the following sequence of "triangles" built up from dots:



(Problem 150)

1.1 List the number of dots in the first five figures.
1.2 How many dots are there in the 100th figure?
1.3 How many dots are there in Fig. *n*?

1.4 Why are numbers of the type $\frac{n(n-1)}{2}$ called triangular numbers?

1.5 Using ideas from above, can you suggest why numbers of the form n^2 are called square numbers?

5.3 THE BINOMIAL COEFFICIENTS

5.3.1 Reflect on the following problems:

1. A set has five elements. How many subsets, each having 3 elements , does it have?
2. You need to select four vertices of a octagon (an 8-sided polygon). In how many ways can this be done?
3. To win the Lotto, you need to select 6 numbers from 48 numbers. In hour many ways can this done? What are the chances of winning the Lotto?
4. How many subsets does a set of n elements have?

Problem 1:

We list all the 3-element subsets:

{a;b;c}	{a;b;d}	{a;b;e}	{a;c;d}	{a;c;e}	{a;d;e}
{b;c;d}	{b;c;e}	{b;d;e}			
{c;d;e}					

The answer is 10.

Now, if the question required us to count the number of words, each having three letters, that can be made from the letters a, b,c, d and e, we would take the first set {a;b;c} and make the words abc,acb,bac,bca,cab,cba – there are six = 3.2.1 = 3! of them – and do the same for the other 9, thus producing 10 x 6 words altogether. But we saw in the last lesson that the number of three lettered words that can be made from five different letters, no repetitions allowed, was, using the box method,

5	4	3

And so is 5 x 4 x 3.

That is:

The number of three element subsets (that is, 10) times the number of ways three elements can be arranged (that is, 6) = = number of three lettered words that can be made from five letters

Hence 10.6 = 5.4.3 so

$$10 = \frac{5.4.3}{1.2.3}$$

Summarising:

A set having 5 elements has $\frac{5.4.3}{1.2.3}$ **three element subsets.**

The argument above can be used to count the number of subsets having **any number of elements!**

For example:

<u>How many five element subsets does a set having 10 elements have?</u>

<u>Answer</u>:

Let N be the number of such subsets. From the previous example,

The number of five element subsets (x) times the number of ways five elements can be arranged (5!)

= number of five lettered words that can be made from ten letters (10 x 9 x 8 x 7 x 6)

That is,

N x 5! = 10 x 9 x 8 x 7 x 6

$$N = \frac{10.9.8.7.6}{1.2.3.4.5} = 252$$

<u>In general, the problem is to determne the number of r-element subsets a set having n elements has</u>.

We have:

5.3.2 Given a set having n elements, <u>the number of subsets having r elements</u> is $\frac{n(n-1)(n-2).....}{1.2.3.....r}$ **where the numerator and the denominator of the fraction in the formula have the same number (namely, r) of factors.**

Let us go back to Problem 2.

<u>Problem 2</u>:
You need to select four vertices of a octagon (an 8-sides polygon). In how many ways can this be done?

Since the order is not important, the count here is the same as the number of subsets, each having four elements, a ten-element set has.

Answer: $\frac{8.7.6.5}{1.2.3.4} = 70$ ways

Problem 3:

To win the Lotto, you need to select 6 numbers from 48 numbers. In hour many ways can this done? What are the chances of winning the Lotto?

Answer: Again, the number of ways is the same as the number of 6-element subsets a set having 48 elements has, which is

$\frac{48.47.46.45.44.43}{1.2.3.4.5.6}$ = 12 271 512

The chances of winning the Lotto are less than 1 in 12 million!

If we multiply the numerator and denominator of the last fraction by $42!$, we obtain that the number of 6-element subsets a set of 48 elements has is:

$$\frac{48.47.46.45.44.43.42!}{1.2.3.4.5.6.42!} = \frac{1.2.3....42.43.44...48}{6!42!} = \frac{48!}{6!42!} = \frac{48!}{6!(48-6)!}$$

an answer which uses only 6 and 48.

In general we have

> **5.3.3 Given a set having n elements, the number of subsets having r elements , where r is one of 0, 1, 2, …n, is** $\frac{n(n-1)(n-2).....}{1.2.3.....r} = \frac{n!}{r!(n-r)!}$
>
> **The notation $\binom{n}{r}$ is used to denote the above expression, that is**
>
> $$\binom{n}{r} = \frac{n(n-1)(n-2).....}{1.2.3.....r} = \frac{n!}{r!(n-r)!}$$

The natural number $\frac{n(n-1)(n-2).....}{1.2.3.....r} = \frac{n!}{r!(n-r)!}$ is a **very important number**, so important that it can be found **on hand-held calculators** in the form **nCr** (read as n choose r) or $\binom{n}{r}$.

Try inputting $\binom{10}{5}$, $\binom{8}{4}$ and $\binom{48}{6}$ in nCr to get 252, 70 and 12 271 512, as we have seen.

Numbers of the form $\binom{n}{r}$ are called **binomial coefficients**.

The case r = 0: A subset having 0 elements? Such a set is called the empty set, which is the set having no elements. It is an acceptable set, and is a subset of **every** set .

The following can be easily verified:

$$1. \binom{n}{0} = 1$$

$$2. \binom{n}{r} = \binom{n}{n-r} \text{ for each r = 0, 1, 2, ...n}$$

5.4 THE NUMBER OF SUBSETS OF A SET

Let us now turn to Problem 4

Problem 4

How many subsets does a set of n elements have?

There is a clever way of counting this number. Suppose we want to count how many subsets {1;2;3;4;5;6;7} has. Draw seven boxes as shown:

Take any subset, like {2;4;7}. **Put 1's in columns 2, 4 and 7, and zeros elsewhere.**

0	1	0	1	0	0	1

On the other hand a string of zeros and ones uniquely identifies a subset. Examples are:

1	1	0	0	1	1	1

{1;2;5;6;7}

0	0	0	0	0	0	0

The empty set

0	0	0	0	0	1	0

{6}

So: **To count the number of subsets, we can count instead, the number of strings of zeros and 1's we can create in our box. But this is easy, if we use the product rule! Each box can be filled in only one of two ways:**

2	2	2	2	2	2	2

There are $2^7 = 128$ **subsets altogether.**

Of course, there is nothing special about the number 7. The same argument can be used for a set having n elements, where n is any natural number.

> **5.4.1 Let a set S have n elements. Then S has** 2^n **subsets.**

5.5 THE EXPANSION OF $(1+x)^n$

A sum having two terms is called a **binomial**. A sum having three terms is called a t**rinomial.** The **binomial theorem** says the following:

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{n-1}x^{n-1} + x^n$$

The numbers $\binom{n}{r}$ occur as coefficients of the powers of *x* in the above formula. Hence they are called the binomial coefficients.

5.6 EXERCISES

1. Verify that $\binom{n}{2} = \dfrac{n(n-1)}{2}$ and $\binom{n}{3} = \dfrac{n(n-1)(n-2)}{1.2.3}$ (Problem 137)

2. How many 3 element subsets does a set having 8 elements have? (Problem 138)

3. How many 98-element subsets does a set having 100 elements have? (Problem 139)

4. Using a reasonable argument, try to explain why $\binom{100}{98} = \binom{100}{2}$ (Problem 140)
5. A set has 1000 elements. How many subsets does it have? (The Eddington number is an estimate of the number of **electrons in the universe.** Its value is less than 2^{264}, so a set having 1000 elements has more electrons than three universes put together!!)(Bin0mial coefficients) (Problem 141)
6. How many whole numbers in the set 1,2,3….99999 have an odd number of odd digits? (Problem 216)

5.7 PROBABILITY

Lesson 4 and much what we did in this Lesson, answered the question "How many?"

To recall:

- If one thing can be done in m ways, and another , in n ways, after the first has been done, in how many ways can both things be done together? (Answer: mn)
- In how many ways can n objects be arranged? (Answer; n!)
- In how many ways can r objects, taken from n objects, be arranged? (Answer: $n(n-1)(n-2)$.... to r factors)
- In how many ways can r objects be selected from n objects, if order does not matter?
  Answer; $\dfrac{n(n-1)(n-2)\text{....to r factors}}{1.2.3.4...r} = {}^nC_r$
- How many subsets does a set of n elements have? (Answer: 2^n)

To calculate the **probability** that some event is likely to happen, one needs to divide the number of "successes" by the number of "total outcomes" , it is at all possible to do so. **The above formulae are very useful in this regard.**

Example

A coin is tossed 6 times and the outcome, namely H or T (that is, a head falls, or a tail falls), is recorded. Thus HHTHTH is a possible outcome. What is the probability that the record shows three heads?

Solution: Each of the six slots can be filled in two ways, so altogether, there are 2^6 = 64 possible strings. (Second counting principle, Lesson 4.1)

For three H's to occur, we have to select three slots from 6, and this can be done in ${}^6C_3 = \dfrac{6.5.4}{1.2.3} = 20$ ways. Hence the probability is $\dfrac{20}{64} = \dfrac{5}{16}$.