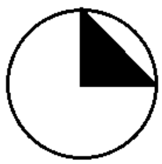
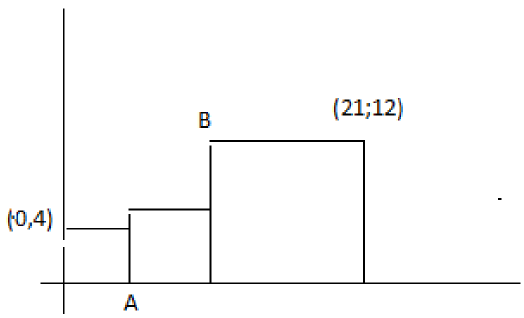
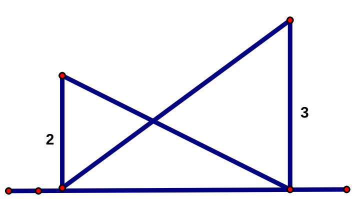
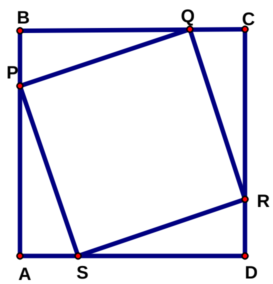
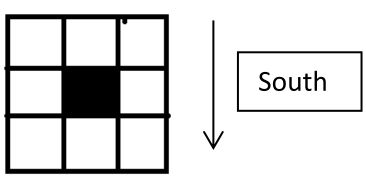
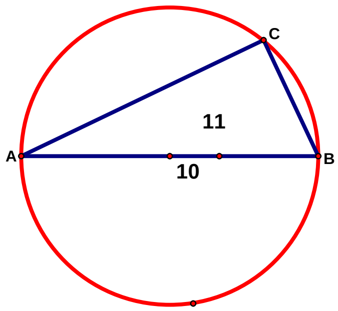
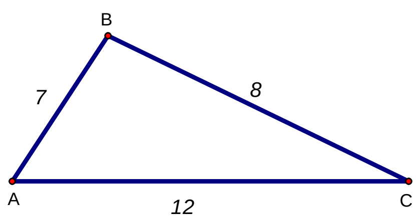
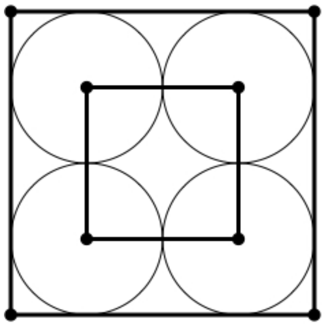
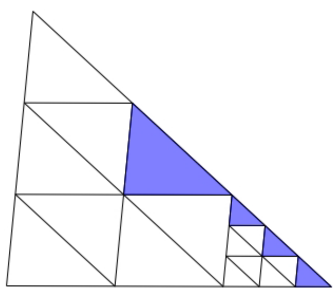
QUESTIONS

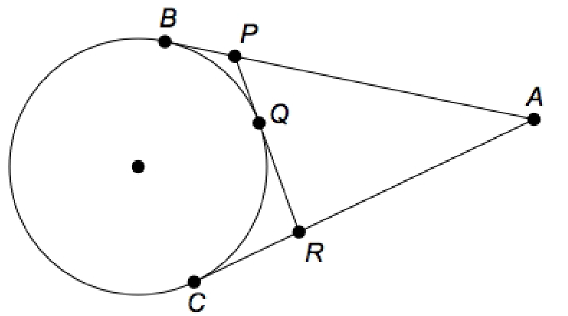
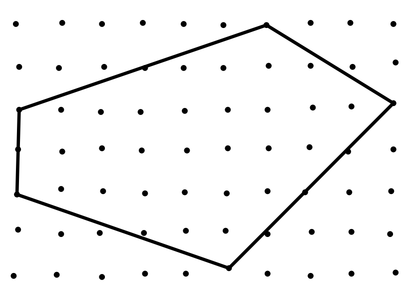
51	4	If m and n are positive integers and $n^2 = 756m$, what is the smallest possible value of m?	L1.1 - 1.3
52	3	 The two perpendicular sides of the right angled triangle form radii of a circle, as shown. Calculate the ratio of the area of the triangle to the area of the circle.	L15
53	4	Two empty containers P and Q have the same volume. Water flows into P at the rate of 4 litres per minute and into Q at the rate of 6 litres per minute. After a certain time, container P can still take another 60 litres, but Q has overflowed by 10 litres. What is the volume of each container?	
54	4	 Three squares are aligned along the x-axis, with coordinates as shown. Determine the shortest distance between the points A and B.	
55	3	The operation * is defined by $x * y = 4x - 3y + xy$ for all real x and y. How many solutions does the equation $x * x = 12$ have?	
56	4	The 24 digit integer 111111111111111111111111 is divided by 1111. How many zeros are there in the quotient?	
57	5	2a + b = c ....(1) a + b + c = 2d...............(2) a + b + c + d = 18 ....(3) for positive integers a, b, c and d, what is the value of c?	
58	6	Determine the number of possible distinct pairs of integers (x, y) for which $(x+2y)^2 + (2x+5y-\frac{1}{2})^2 \leq 2$.	
59	4	The average age of teachers at an institution is 35 years, and the average age of its professors is 50 years . If the average age of teachers and professors together is 40 years, what is the ratio of the number of teachers to the number of professors?	L1.6
60	4	Calculate 123^2x129 - 124x125x126 without a calculator.	

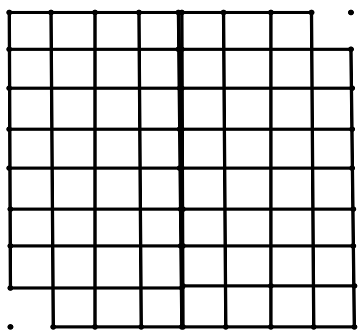
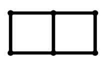
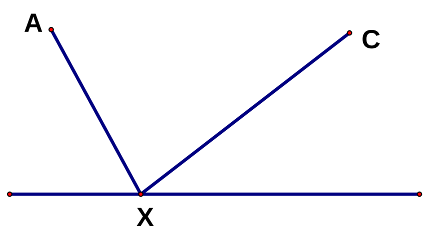
61	6	Sixty entrants pitch up for a knockout tennis tournament. (In a knockout, the loser of a game is eliminated). Any number of “standbys” are allowed at each stage (for example, if there were 47 players, we could have 12 games and 23 standbys). At each stage, the players for the next round are chosen from the winners of the previous round and the standbys. Of course, there are no draws. Every one plays in at least one game. How many games are played altogether? Explain why, no matter how many standbys there are in each round, the number of games played is always the same.	
62	2	(a)What is the sum of: $1 + 2 + 3 + . . . + (n – 2) + (n – 1) + n$? (b) of $1 + 2 + 3 + . . . + 60$?	L 8.3
63	3	Discuss the parity of product, difference and sum of two integers with reference to the parity of each of the two integers. (Parity is the “evenness” or “oddness” of a number. Thus 34 has even parity and 27 has odd parity).	L3.8
64	4	If m and n are both odd integers, then which of the following numbers must be even? (a) mn (b) $m^2n + 2$ (c) $m + n + 1$ (d) $2m + 3n + 5$ (e) $2m + n$	L3.8
66	5	1. Find all the different ways in which the number 2000 can be expressed as the difference of the squares of two positive integers. 2. Find all right angled triangles that contain 12 as a side that is, not the hypotenuse.	L 7
69	3	What is the last digit of 3^{2014}?	
72	5	Find the last two digits of 6^{2014}.	
73	6	What is the second last digit when the product 1 x 3 x 5 x 7 x . . . x 99 is written as a number?	
74		Two vertical sticks in the ground have lengths 2 metres and 3 metres. The top of each stick is joined to the bottom of the other, by means of two strings. The strings meet at appoint P. What is the height of P above the ground? 	L15
75	4	If you write the integers 2, 3, 4, 6, 8 in every possible order to form 5-digit numbers, how many of these numbers will be divisible by 11?	L1.5

76	5	Find the value of $k + l$ if k and l are positive integers and $k + l + kl = 54$.	L1.1
77	2	What is the smallest number that leaves a remainder of 3 when divided by 10 and leaves a remainder of 4 when divided by 13?	
78	5	How many positive integers n are there such that $n + 3$ is a factor of $n^2 + 7$?	L1.3
81	321	 Points P, Q, R, and S are marked on the sides of square ABCD so that each side is divided in the ratio 2 : 1, and therefore PQRS is a square. Calculate the ratio of the area of PQRS to the area of ABCD.	
82	3	The natural numbers are written in seven columns 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 … A square is drawn around a certain block of four numbers and the sum of those four numbers is 312. What is the number in the top left square? (In the example above, the sum is 28 and the top left square is 3).	
83	5	A block of eight flats with a stairway in the middle is situated on the top floor of a building. The positions of the flats are shown on the sketch below. There are windows on all sides of the building which afford good views. Twice as many people have a southward view as there are people who can look eastward. Those with a westward view number only one third of those who can look south, while the few who have a northward view, number only half of those who can look east. Altogether 20 people live in the eight flats. How many occupy each flat, given that no flat is vacant? Explain your answer and show the number of occupants on a sketch. 	

84	5	 AB is the diameter of the semicircle and AB = 10. If the area of the ΔABC is 11, find the perimeter of ΔABC.	
86	5	Fareeda would like to become an Olympic sprinter. Her younger sister Sumayya would rather play football, but helps Fareeda by racing against her. When they tried the 100 metre dash, Fareeda crossed the winning line when Sumayya was still 20 metres short of it. Fareeda wanted something more challenging, so it was agreed that Fareeda would start 20 metres behind the starting line. They both ran exactly the same speeds as in the first race. Where were Fareeda and Summaya when the winning line was crossed by whoever arrived at it first?	
87	6	Aster, Baster, and Caster are three villages, as shown in the diagram below, where the straight lines represent the only roads joining the villages. The figures give the distances in kilometres between villages.  A new fire station is to be built to serve all three villages. It is to be on a roadside at such a position that the greatest distance that the fire-engine has to travel along the roads in an emergency at one of the villages, is a small as possible. Where should the fire station be positioned? Locate your point on the diagram, and explain why no other position is satisfactory.	
88	3	A child's age, increased by 3, gives a perfect square, and when decreased by 3 the age is the (positive) square root of that perfect square. How old is the child?	
89	2	Calculate the value of: $\dfrac{2^1+2^0+2^{-1}}{2^{-2}+2^{-3}+2^{-4}}$	
90	3	If $10^{101}-1$ is written out in full, find the sum of the digits of this number.	
91	3	In the diagram, the congruent circles are tangent to the large square and each other as shown; and their centres are the vertices of the small square. The area of the small square is 4. Find the area of the large square.	

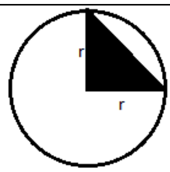
			
92	3	I wrote down the integers 25, 26, 27, . . . , 208. How many **digits** did I write down?	
93	3	If the number A1234567B is divisible by 45, determine the value of A + B.	L1.5
94	3	Find the value of $2013^2 - 2(2000)(2013) + 2000^2$	
95	4	Calculate the value of 2013−2009+2005−2001+1997−1993+..+29−25.	L8.5
96	3	In the diagram, the sides of the largest triangle are divided into three equal parts to produce smaller congruent triangles as shown. This process is repeated for the smaller triangle on the bottom right. If the area of the largest triangle is 81, what is the total area of the shaded triangles? 	
97	3	A bag contains 65 marbles of the same size. There are 20 red ones, 20 green ones, 20 blue ones, and another 5 that are either yellow or white. Lindiwe removes marbles from the bag without looking. What is the smallest number of marbles that she must remove to ensure that she has 10 of the same colour?	
98	5	Determine the number of pairs (x; y) of integer solutions for $2^{2x} - 3^{2y} = 55$	L2.5
100	4	Two tangents are drawn to a circle from a point A, which lies outside the circle; they touch the circle at points B and C respectively. A third tangent intersects AB in P and AC in R, and touches the circle at Q. If AB = 20 and PQ = 3, find the perimeter of triangle APR.	L15

			
102	5	Jack, John and James are identical triplets. It is impossible to distinguish them by appearance. Jack and John always tell the truth, but James always lies — everything he says is false. You know that the triplets are between 20 and 30 years old, 20 and 30 included. One day you meet two of the triplets and ask them how old they are. A says 'We are between 20 and 29 years old, 20 and 29 included'. B makes the following statements: 'We are between 21 and 30 years old, 21 and 30 included' and 'One of us present is lying'. How old are they?	
103	5	John takes 300 steps to walk from point A to point B in a flat field. Each step is of length $\frac{1}{\sqrt{2}}$ meters, and he makes a 90° turn after every step except after the last one. He makes 99 left turns and 200 right turns in total. He stops at point B. What is the maximum possible distance from A to B?	
106	5	Erica noted that a train to Muizenberg took 8 minutes to pass her. A train in the opposite direction to Cape Town took 12 minutes to pass her. The trains took 9 minutes to pass each other. Assuming each train maintained a constant speed, and given that the train to Cape Town was 150m long, what was the length of the train to Muizenberg?	
108	4	**Pegs are nailed into a board 1cm apart as shown in the diagram. An elastic band is stretched over five pegs as shown. What is the area of the pentagon so formed?** 	

109	6	 Two opposite corners of a eight by eight grid are removed, so 62 squares are left. You have thirty one 1 x 2 dominos. Place the dominos on the grid in such a way that all 62 squares on the grid are occupied. 	
110	2	How large is a billion?Suppose I try to count to a billion, that is 1 000 000 000. If I count one every second, without stopping to rest, how long would it take me?	
112	4	The product of the ages of a group of children whose ages are all between 12 and 20 is 10 584 000. How many children are there in the group?	L1.1 – L1.4
113	6	Two points A and C lie on the same side of a straight line. Find a point X on the line such that the sum AX + XC is as small as possible. 	
114	6	The square ABCD has sides of length 6 units. M is the midpoint of AB and P is a variable point on BC. Find the smallest value of DP + PM.	
115	5	ABCD is a rectangle, and P is an arbitrary point in its interior. Determine PA in terms of PB, PC and PD.	
116	5	Between 12.00 and 13.00, there are two times when the hands of a clock are exactly at right angles. How many minutes apart are these two times?	
118		The 64 squares of a chessboard are populated with 0 and 1. Prove that amongst the 18 row/column/ diagonal sums, at least three are equal.	

SOLUTIONS

51	$n^2 = 2.2.3.3.3.7m$. For the LHS to be a perfect square, the exponents in the prime factorisation factors on the right must be even. So m must contain a 3 and a 7 in its factorization. Least m is 3x7 =21

52	 The area of the triangle is $\frac{1}{2}$ base times height = $\frac{1}{2}r^2$ while the area of the circle is πr^2. The ratio is $\frac{1}{2}r^2 : \boldsymbol{\pi} r^2 = 1 : 2\boldsymbol{\pi}$.
53	**<u>Method 1</u>**: The excess of Q over P, namely, 70 litres, was achieved at the rate of 6 – 4 = 2 litres per minute, so the water must have been flowing for 35 minutes. 35 x 4 + 60 (or 35 x 6 – 10) = 200 **<u>Method 2</u>**: After t minutes, volume of water in P (resp. q) is 4t (resp. 6t). Let V be the volume of the each container in litres. Then V – 4t = 60 and 6t – V = 10. Add: 2t = 70, so t = 35. So V = 60 + 4t = 60 + 4(35) = 200
54	The shortest distance is the length of the straight line AB. AB is the hypotenuse of a right angled triangle. One side is the y-coordinate of B, which is the y-coordinate of (21 ; 12), hence 12. (x-coordinate of B) + 12 = 21, so x co-ordinate of B = 9 The other side of the right-angled triangle is (the x-coordinate of B) – 4 = 9 – 4 = 5; Hence $AB^2 = 5^2 + 12^2 = 169$. AB = 13.
55	$x * x = 4x - 3x + x^2 = 12$ $x^2 + x - 12 = 0$ $(x+4)(x-3) = 0$ **Two s**olutions.
56	Group the 1's into fours. The quotient has 6 ones, and there are 3 zeros between every pair of consecutive 1's. Answer: 5 x 3 = **15** **OR** The quotient has 6 ones, the rest being zeros. The quotient has 24 – 3 digits. So there are 24 – 3 – 6 = 15 zeros.
57	Substitute (2) in (3). 2d + d = 18, so d = 6. So from (3), a + b + c = 12……………..(4) Substitute (1) in (4). a + b + (2a + b) = 12 and

	3a + 2b = 12……………………………………(5) **3a = 2(6 – b) is positive and both a and b and are integers.** So 6 – b is a multiple of 3. But 6 is a multiple of 3, hence b is a multiple of 3, that b = 3, 6, 9,….Also, since 6 – b > 0, we have b = 1,2,3,4 or 5. Hence b = 3. Substitute in (5): 3a + 6 = 12, so a =2. From (4), c = 7. **<u>Alternate</u>:** Solve for a and b in terms of c to get a = 2(c – 6), b = 3(8 – c). Since a and b are positive integers, 6 < c < 8. That is, c = 7
58	Suppose $(x;y)$ is a solution of $(x+2y)^2+(2x+5y-\frac{1}{2})^2 \le 2$ ……………………(1). Then $(x+2y)^2 \le (x+2y)^2+(2x+5y-\frac{1}{2})^2 \le 2$, and, since x + 2y is an integer, x + 2y = 0 or x+ 2y = 1 or x + 2y = -1 **In general, x + 2y = a where a = -1, 0 or 1.** Then x = a – 2y. Substitute in (1): $a^2+(2a-4y+5y-\frac{1}{2})^2 \le 2$ $(y+2a-\frac{1}{2})^2 \le 2-a^2$ It follows that $-\sqrt{2-a^2}-(2a-\frac{1}{2}) \le y \le \sqrt{2-a^2}-(2a-\frac{1}{2}),\ a=-1,0,1$ and (x ; y) = (a-2y ; y). We have $a=-1 \Rightarrow -1-(-\frac{5}{2}) \le y \le 1-(-\frac{5}{2}) \Rightarrow \frac{3}{2} \le y \le \frac{7}{2} \Rightarrow y=2$ or 3 $\Rightarrow (x;y)=(-1-4;2)=(-5;2)$ or $(-1-6;3)=(-7;3)$ $a=0 \Rightarrow -\sqrt{2}+\frac{1}{2} \le y \le \sqrt{2}+\frac{1}{2} \Rightarrow -0{,}9 \le y \le 1{,}9 \Rightarrow 0$ or 1 $(x;y)=(0;0)$ or $(-2;1)$ $a=1 \Rightarrow -1-\frac{3}{2} \le y \le 1-\frac{3}{2} \Rightarrow y=-2$ or -1 $\Rightarrow (x;y)=(5;-2)$ or $(3;-1)$ There are six solutions: $(-5;2)$, $(-7;3)$, $(0;0)$, $(-2;1)$, $(5;-2)$, $(3;-1)$
59	Let x and y be the number of teachers and professors respectively. We need to find $\frac{x}{y}$. Since 35 is the average age of the teachers, their total age is 35x. Likewise, total age of professors is 50y. So the total combined age is 35x + 50y. But the average age of all take together is 40. Hence

$$35x + 50y = 40(x + y)$$

$$5x = 10y$$

$$\frac{x}{y} = 2.$$

60 Notice that it is easy to multiply by 125, since 125 x 8 = 1000 and 125 x 4 = 500. **So let x = 125.**
Then
123^2x129 - 124x125x126
= $(x - 2)^2(x + 4)-(x - 1)x(x + 1)$
=$x^3 - 12x + 16 - x^3 + x$ = 16 – 11x = 16 – 11.125
= 16 – 1375 = - 1359

61

Players	Standbys	Games
28	32	14
46	0	23
16	7	8
12	3	6
8	1	4
4	1	2
2	1	1
2	0	1

We could have, for example, the situation above. The total number of games played is 14 + 23 + 8 + 6 + 4 + 2 + 1 + 1 = 59.

Explanation : Each game has exactly one loser, and everyone who lost, **lost exactly one game. So there is 1-1 correspondence between "games" and "losers" .** That is, the number of games is exactly the same as the number of players who lost. But there is exactly one player who did not lose any game, namely, the winner of the tournament. So there are 59 games.

62 $\frac{n(n+1)}{2}$ and
$\frac{60(60+1)}{2}$ = 1830

63 Clearly the sum of two integers is even **if and only if** either both are even or both are odd. The product is odd if and only if both the numbers are odd. (If either of the numbers is even, the product is even).

If you have done Lesson 5, there is another way of seeing the above.

In mod 2, even numbers are congruent to 0 and odd numbers are congruent to 1. So in mod 2,
even + even = $0+0 \equiv 0$ is even
and odd + odd = $1+1 \equiv 0$ is even. Since $0+1 \equiv 1$, the sum of numbers having opposite parity is odd.
Now $xy \equiv_2 1$ forces $x \equiv_2 1$ and $y \equiv_2 1$. that is, both x and y are odd if their product is odd.

In mod 2, the equation become:

64

(a) mn $\equiv 1.1 = 1$
(b) $m^2n \equiv 1.1.1 = 1$
(c) m + n + 1 = 1+1+1=1
(d) 2m+3n+5 $\equiv 0.1 + 1.1 + 1 = 0$
(e) 2m+n=0.1+1=1

So only (d) is even.

66

1. We write 2000 as a product ab of two numbers having the same parity, and use the identity $ab + \left(\frac{a-b}{2}\right)^2 = \left(\frac{a+b}{2}\right)^2$

ab (a < b)	½ (b – a)	½(a + b)
2.1000	449	501
4.500.	248	252
8.250	121	129
10.200	95	105
20.100	40	60
40.50	5	45

So $2000 = 501^2 - 449^2$
$= 252^2 - 248^2$
$= 129^2 - 121^2$
$= 105^2 - 95^2$
$= 60^2 - 40^2$
$= 45^2 - 5^2$
and these are only solutions.

2. $12^2 = 144 = 2.72 = 4.36 = 6.24 = 8.18$ are the only factorisations of 144 into two numbers having the same parity.

From $12^2 = 2.72$ we deduce $12^2 + \left(\frac{72-2}{2}\right) = \left(\frac{72+2}{2}\right)^2$ so (12,35,37) is a Pythagorean triple.

The others are
(12, 16, 20), (12, 9, 15), (12, 5, 13). There are four such triangles.

69

The last digit of 3^n, for n = 1,2,3,4,5,6,7,8 is
3, 9, 7, 1, 3, 9, 7, 1. So if n = 4, 8, 12, 16,…..the last digit is 1. Since 2012 = 4.503 is a multiple of 4, the last digit of 3^{2012} is 1. The last digit of $3^{2014} = 3^{2012}.3^2 = 3^{2012}.9$ is therefore 9.

72

$6^2 = 36.$

$6^4 = 36.36 = 1296$
ends in 96.
$6^6 = 36(1296)$, and since 36(96) = 3456 ends in 56.so does 6^6.
6^8 = 36(…56) has the same last two digits of 36(56) = 2016, that is the last two digits of 6^8 are 16.
Likewise 6^{10} ends in 76 and 6^{12} ends in 36. Summarising the last two digits of
$6^2, 6^4, 6^6, 6^8, 6^{10}, 6^{12}, 6^{14}, 6^{16}, 6^{18}, 6^{20}$,…… are
36, 96, 56, 16, 76, 36, 96, 56, 16, 76,….
If the index (=power, exponent) is 10, 20, 30, …., the last two digits are 76. Hence when the index is 2010, the last two digits are 76, if 2012 they are 36 and for 2014, they are 96.

73 The number is odd and also an odd multiple of 25, so ends in 25 or 75. (The even multiples of 25 end in 50 or 00). If it is of the first type, it is congruent to 1 mod 4, and if it is of the second type, it is congruent to 3 mod 4, which is also (-1) mod 4.

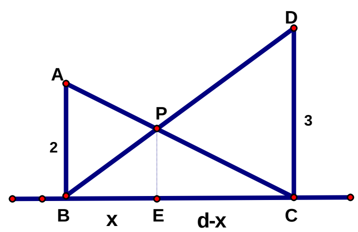
So let reduce the number, mod 4.

1 x 3 x 5 x 7 x . . . x 99

=(1 x 5 x 9 x….97) x (3 x 7 x ….99)

$\equiv_4 1^{25}(-1)^{25} \equiv_4 (-1) \equiv_4 3.$

The number ends in 75, so the second last digit is 7.



The distance BC between the poles is not given. Call it d.

We need to calculate PE, where PE, like AB and DC, is vertical to the ground. Let E be x units from base B of the shorter of the sticks.

ΔCPE is similar to ΔCAB . So

$$\frac{PE}{AB}=\frac{CE}{CB}=\frac{d-x}{d} \text{ ....(1)}$$

Also ΔBPE is similar to ΔBDC

$$\frac{PE}{DC}=\frac{BE}{BC}=\frac{x}{d} \text{ ....(2)}$$

Adding (1) and (2), we get:

$$\frac{PE}{AB}+\frac{PE}{DC}=\frac{d-x+x}{d}=1$$

$$PE\left(\frac{1}{2}+\frac{1}{3}\right)=1$$

$$PE\left(\frac{5}{6}\right)=1$$

$$PE=1{,}2\text{m}$$

Note that the height is the same no matter how far apart the sticks are, a surprising result!

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75 Let x be the sum of the 2nd and 4th digits of a possible answer to the problem, and y the sum of the 1st, 3rd and 5th digits. We have that x + y = 2 +3 + 4 + 6 + 8 = 23 and the difference between x and y is a multiple of 11, since the number is a multiple of 11. By trial and error, we see only the numbers 6 and 17 work. x cannot be 17 – the highest value for x is 6 + 8 = 14 – so x = 6. That is x = 2 + 4 (only).

So we have:

?	2	?	4	?

or

?	4	?	2	?

The number 3, 6 and 8 fill the blanks in each case. Regardless of their order, all these numbers are divisible by 11, and no other. There are 6 + 6 = 12 of them altogether.

---

76 $(k+1)(l+1)=k+l+kl+1=54+1=55$. So k + 1 is a factor of 55. It is one of 1, 5, 11 or 55. If it is 1 or 55, we have that either k or l is 0, a contradiction.

Since, so (k+1, l+1) = (5, 11) or (11, 5) and k + l +2 = 15.

	Hence k + l = 14.
77	The number is in 3, 13, 23, 33, 43, ….as well as 4, 17, 30, 43, … Answer is 43.
78	We need to determine the positive integers n for which $\frac{n^2+7}{n+3}$ is an integer. $\frac{n^2+7}{n+3} = \frac{n(n+3)-3(n+3)+16}{n+3}$ $= n-3+\frac{16}{n+3}$ **OR** (Long divide n^2+7 by n + 3 to obtain quotient n – 3 and remainder 16. Then $n^2+7=(n-3)(n+3)+16$. Divide both sides by $n+3$). The left hand side is an integer if and only if $\frac{16}{n+3}$ is an integer. That is, $n+3$ is a factor of 16, with $n > 0$. $n+3 \in \{4,8,16\}$ giving three values for n, namely, 1, 5 and 13. (**Check**: $\frac{8}{4}$, $\frac{32}{8}$ & $\frac{176}{16}$ are all integers).
318	So by Pythagoras, $PQ^2 = BQ^2 + BP^2 = (2x)^2 + (x)^2 = 5x^2$, but PQ^2 is the area of square *PQRS*. The area of *ABCD* is therefore $(2x+x)^2 = 9x^2$. So $\frac{\text{Area of PQRS}}{\text{Area of ABCD}} = \frac{5x^2}{9x^2} = \frac{5}{9}$
82	Lex x be the number. Then he four numbers are x x+1 x +7 x+8 So the sum is 4x + 16 which is equal to 312. 4x = 296 and x = 74

83

a	b	c
h	**X**	d
g	f	e

Let us use N, E, W, S to denote the directions, and name the rooms as indicated. Then

N = a + b + c,

E = c + d + e

W = a + h + g and

S = g + f + e,

where each of N, E, W and S is at least 3, all rooms are occupied.

From the given information,

2E = S,

S = 3W, and

2N = E. So

S = 3W = 2E = 4N. Now N $\geq$3, but N $\geq$ 4 is impossible since then S$\geq$16 so the total number of people is at least 16 + 5 = 21, and we have only 20 people altogether.

1	1	1
	X	

So N = 3, making

a = b = c = 1, and

S = 12, W = 4,

E = 6.

W =4

1	1	1
	X	
	f	

E = 6

S = 12

But

E + W = 10

E + W + 1 + f =total number of people = 20.

So 1 + f =10, and f = 9

Since S = 12 and f = 9, we have g + e = 3 so g is 1 or 2. Both values produce solutions.

1	1	1
2	**x**	3
1	9	2

1	1	1
1	**x**	4
2	**9**	1

84

The angle in a semicircle is a right angle. So $\angle C = 90^\circ$.

Let a = BC, b = AC. We need to determine the perimeter, that is, a + b + 10, so we need to find a + b.

By Pythagoras

$a^2 + b^2 = 100$. Also the area of the triangle is 11. That is,

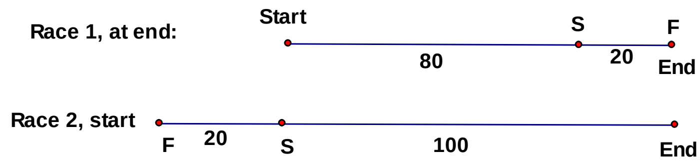
$\frac{1}{2}ab = 11$, or ab =22.

The above two equations, and the fact that we need to calculate $a(a + b)^2$

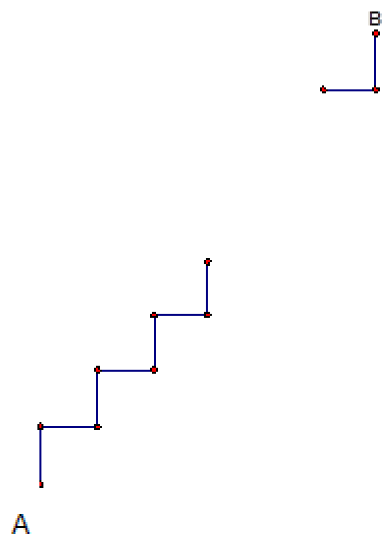
$(a + b)^2 = a^2 + b^2 + 2ab = 100 + 2(22) = 144$

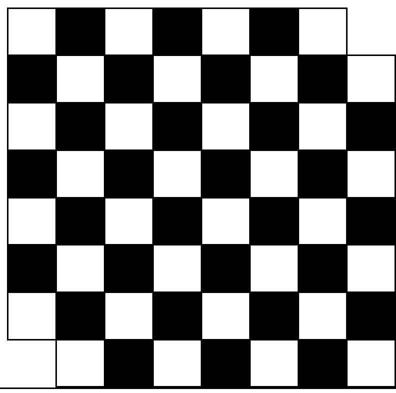
Hence a + b = 12 and the perimeter a + b + 10 = 22 units.

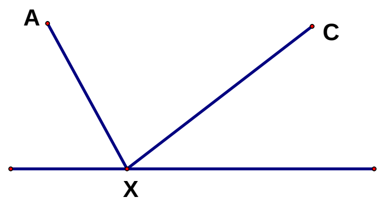
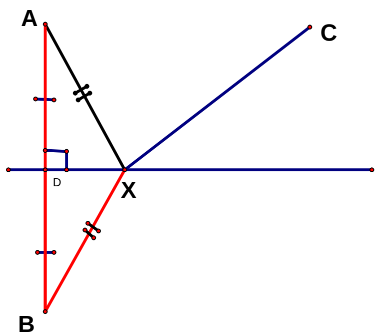
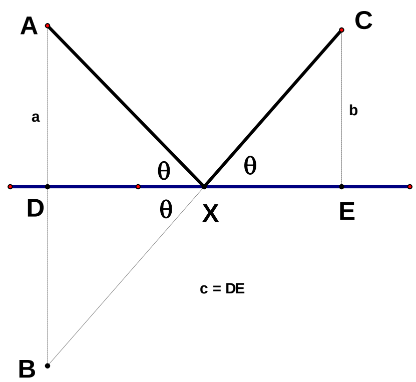
86

	 Since Sumayya covers 80 when Fareeda covers 100, Sumayya's speed is $\frac{4}{5}$th the speed of Fareeda's. If, on the second run, Sumayya finishes first, she would have run 100, while Fareeda would have run $\frac{5}{4}(100)=125$ metres. But that would have made Fareeda first, a contradiction. So Fareeda was first and Sumayya ran $\frac{4}{5}(120)$ = 96 metres, 4 metres behind Fareeda at the winning post.
87	The location F of the station is along BC, with BF = ½. The distances are then ½, 7½ and 7½ , the maximum distance being 7½. The max for any point along BA is greater than 8. For a point G on AB to better 7½, AG has to be less than ½,but then max is greater than 11½ . For any G on BC, AG less than or greater than ½ increases the max beyond 7½. Suppose the point in question is along AC. For example, if AX = 4 and XC = 8, the maximum is the maximum of { 4, 8, 4+ 7 = 11} which is 11. We can reduce 11 by moving X closer to A. For example, if AX = 3, the maximum distance is the maximum of {3, 9, 10), which is 10. Evidently, the best point on AC wuld be that point X where XC = XA + 7. So X will exactly half way on the bent line BAC. Then 7 + AX = $\frac{1}{2}(7+12)=\frac{19}{2}$. So the best point X on AC is such that AX = 2½, giving a maximum distance of 9½. The "smallest of the largest distances" on each of the other two lines, are, similarly, $\frac{1}{2}(8+12)=10$ & $\frac{1}{2}(7+8)=7\frac{1}{2}$. So the last is best; the point X on BC with BX = ½ ha greatest distance equal to the maximum of { ½, 7½, 7½} = 7½
88	Let x be the age of the child. Then $x+3=a^2$ and $x-3=a$ where a is a positive integer. So $a^2=(a+3)+3$, $a^2-a-6=0$, $a=3$ or -2. The child's age is x = a + 3 = 6.
89	$\frac{2^1+2^0+2^{-1}}{2^{-2}+2^{-3}+2^{-4}}=\frac{2^4(2^1+2^0+2^{-1})}{2^4(2^{-2}+2^{-3}+2^{-4})}=\frac{2^5+2^4+2^3}{2^2+2+1}=8$ **<u>Alternate</u>**: Note that the indices in the numerator are consecutive numbers, and the same holds for the denominator. The question is therefore: what must the bottom be multiplied by to get the top? Answer 2^3 = 8.

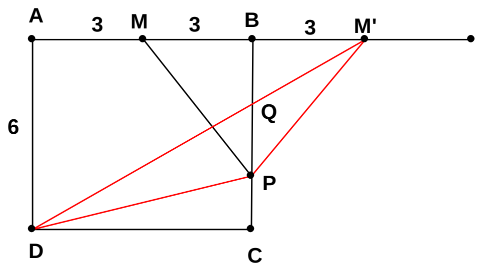
90	$10^2 - 1 = 99$ has 2 digits. Likewise $10^{101} - 1$ has 101 digits, all of which are 9's. The sum is 101 x 9 = 909.
91	Each side of the small square has length 2, so the radius of each circle is 1. A side of the large square has length 2 x diameter = 4. Its area is 16.
92	From 25 to 99, there are 99-24=75 numbers and 150 digits. From 100 to 208 there 208 – 99 = 109 numbers and 327 digits. So there are 327 + 150 = 477 digits altogether.
93	The number is a multiple of 5 and so ends in 0 or 5. So B is 0 or 5. The sum of the digits is divisible by 9, so A + B + 28 is divisible by 9. So A + B is among 8, 17, 26,…But A + B is less than or equal to 9 +5 = 14. Hence A + B = 8.
94	**<u>Method 1</u>**: An examination of the expression reveals that it is in the form $a^2 - 2ab + b^2 = (a-b)^2$ so the answer is $(2013-2000)^2 = 13^2 = 169$ **<u>Method 2</u>**: Let a = 2000. Then the expression is equal to $(a+13)^2 - 2a(a+13) + a^2$ $= a^2 + 26a + 169 - 2a^2 - 26a + a^2$ $= 169$
95	There are as many terms in this expression as there are numbers in 2012, 2008, 2004,…28, 24. So there are 503 –(6-1) = 498 terms. Pair and sum to get 4 + 4 + 4 +…4 , 249 times. Answer is 996.
96	Each triangle is divided into 9 equal triangles. The largest shaded triangle has area $\frac{1}{9}(81) = 9$. The three smaller triangles have a combined area of $\left(\frac{3}{9}\right)\left(\frac{1}{9}\right)(81) = 3.$ Answer is 9 + 3 = 12.
97	The worst possible case occurs when the first colours removed are: 5 (yellow or white), 9 of each of the blue, green and red. The next one will ensure that she has ten of the same colour. The answer is 5 + 9 + 9 + 9 + 1=33.
98	$2^{2x} - 3^{2y} = (2^x)^2 - (3^y)^2 = (2^x - 3^y)(2^x + 3^y) = 55$. The first factor is the smaller than the second so we have $2^x - 3^y = 1$ and $2^x + 3^y = 55$ OR $2^x - 3^y = 5$ and $2^x + 3^y = 11$ From the first two, we obtain, by adding: $2^{x+1} = 56$ which is not possible since 56 is not a power of 2. From the second pair of equations, $2^{x+1} = 16,\ x+1 = 4,\ x = 3,\ 3^y = 11-8,\ y = 1$ So $(x;y)$ = (3 ; 1) is the only solution. Answer: 1 solution
100	We know PB = PQ and RQ = RC. The perimeter is AP + PR + AR = AP + PQ + QR + AR = AP + PB + RQ + AR = AB + AC =2 AB = 40

102	If B is the liar, he speaks the truth when he says that one of them is lying. So B cannot be the liar. He speaks the truth so (a) A is the liar (b) If x is their common age, $21 \le x \le 30$. Since A lies, it is not true that $20 \le x \le 29$ So x = 30
103	After the 99 left turns are utilized, John can make only right turns, and every four such turns will bring him back to starting point. Hence to maximise the distance from A to B, every left turn should take him closer to B. The first step is neither right nor left, but the second should be right, as the goal is to use the lefts to maximise AB. This will require 100 right turns. It does not matter how he uses the remaining 100 right turns. The distance AB is made up of 100 lengths, each of which is the hypotenuse of a right angled isosceles triangle having side $\frac{1}{\sqrt{2}}$. So maximum distance is 100x 1 = 100. (Note that if the second turn is left, we will have 99 triangles).  Then $AB = 100\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} = 100$
106	Let x and y (=150) be the lengths (in metres)of the Muizenberg and Cape Town trains respectively. In 1 minute, $\frac{x}{8}$ metres of the train passes Erica, so the Muizenburg train travels at $\frac{x}{8}$ metres/min. Likewise the Cape Town train travels at $\frac{y}{12}$ metres per minute. Let A refer to the front of one train, and B the front of the other. When travelling in opposite directions, AB, the distance between A and B, is 0 at the start and x + y at the end. A and B move away from each other at a speed of $\frac{x}{8} + \frac{y}{12}$ metres per second, and in 9 minutes cover the distance x + y. Hence, $9\left(\frac{x}{8} + \frac{y}{12}\right) = x + y$

	Substituting y = 150, we obtain $x = 8.\frac{3y}{12} = \frac{24.150}{12} = 300\,\text{metres}$
108	Area of pentagon + sum of the areas of the four triangles = Area of rectangle Using the ½ base times height formula, the sum of the areas of the triangles is ½ (6 x 2 + 3 x 2 +4 x 4 +5 x 2) = 22 sq. cm. Area of rectangle = 9 x 6 = 54 $\therefore$ Area of pentagon = 54 – 22 = 32 sq.cm
109	 The trick is to paint the grid as if it were a chessboard, as in the diagram above. Note that the removed squares are both black, so we have 30 black and 32 white squares. But the 31 dominos will cover 31 black and 31 white squares! We conclude that it is impossible to place the dominos as required.
110	It would take a billion seconds. But how long is that? Let us do the calculation: $1000\,000\,000\,\text{seconds} = \frac{10^9}{60}\,\text{minutes} = \frac{10^9}{60.60}\,\text{hours} = \frac{10^9}{60.60.24}\,\text{days} = \frac{10^9}{60.60.24.365}\,\text{years}$ ≈ 32 years!
112	10584000=1000.10584=10.10.10.4.2646 = 2.5.2.5.2.5.4.2.1323 =2.5.2.5.2.5.2.2.2. 9.147=2.5.2.5.2.5.2.2.2.3.3.3.7.7 $= 2^6.3^3.5^3.7^2$ The product contains two 7,s. These can come only from 14. So that are two 14's. The product of the other numbers is $\frac{2^6.3^3.5^3.7^2}{14.14} = 2^4.3^3.5^3 = 2^4(15^3) = 16(15)^3$ So the number is 14.14.15.15.15.16 making six children altogether. (Another possibility is 12.14.14.15.15.20.) **<u>Alternate:</u>** The product of 5 numbers from the list is less than $20^5 = 3\,200000 < 10\,584000$ and the product of 7 numbers is greater than $11^7 = 121.121.121.11 > 11.100.100.100 > 10\,584\,000$ So the answer must be 6.

113	Reflect A across the line and call the image, B. Then AX + XC = BX + XC which, by the triangle inequality, is greater or equal to BC.  
	 So BXC must be a straight line for the sum to be as small as possible. If $\angle AXD = \theta$, then since triangles AXD and BXD are congruent, $\angle BXD = \theta = \angle CXE$, that is, “the angle of incidence = the angle of reflection” The answer to the question is to choose a point on the line so that the angle of incidence = the angle of reflection.

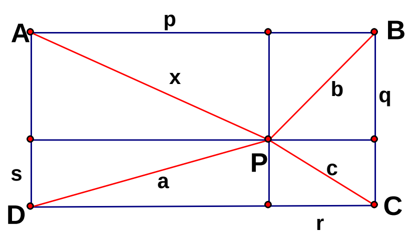
114



Reflect M across BC to M'. For any point P on BC, DP + PM = DP + PM' $\geq$ DM', so P must be chosen as the point Q of intersection of BC and DM'. Then

DQ + QM = DQ + DM' = DM' = $\sqrt{DA^2 + (AM')^2} = \sqrt{6^2 + 9^2} = \sqrt{117}$

115



Let PA, PB, PC and PD be equal to x, a, b and c respectively. We need to determine x in terms of a, b and c. Draw lines through P parallel to the sides of the rectangle. Name the segments p, q, r and s as shown. We have

$x^2 = p^2 + q^2$

$b^2 = q^2 + r^2$

$c^2 = r^2 + s^2$

$a^2 = p^2 + s^2$

So $$x^2 + c^2 = a^2 + b^2 = p^2 + q^2 + r^2 + s^2$$
$$x = \sqrt{a^2 + b^2 - c^2}$$

116

Observe that the hour hand moves through 5 minutes during the same time that the minute hand moves through 60 minutes. So the minute travels 12 times faster than the hour hand.

When the minute hand moves through x minutes, the hour hand covers $\frac{x}{12}$ minutes, so the number of minutes separating them is $x - \frac{x}{12} = \frac{11x}{12}$. But each minute is $\frac{1}{60}\left(360^\circ\right) = 6^\circ$ so the angle between them (when the minute hand moves through x minutes) is $\left(\frac{11x}{12}\right)6^\circ = \frac{11x}{2}$ degrees. When they are at right angles to each other we have either $\frac{11x}{2} = 90$ or $\frac{11x}{2} = 270$, so x is either $\frac{180}{11}$ or $\frac{540}{11}$ minutes. The number of minutes between these two times is $\frac{360}{11} = 32\frac{8}{11}$ minutes.

118

0	0	0	0	0	0	0	0

We prove the assertion by contradiction, that we **assume** it is not true, and eventually arrive at something known to be false. And then we can conclude that our assumption is false, proving the truth of the assertion.
So let us assume that no three of the eighteen sums are equal.
Note first that the only possible sums are 0,1,2,3,4,5,6,7 and 8.
None of these 9 numbers occurs as a sum more than twice.
Indeed, as there are 18 numbers, each sum occurs **exactly twice.**
The sum 0 occurs only when all eight entries are 0. If it is a diagonal sum, no row or column can have sum 8, so both diagonals will have to have sum 8, which is impossible.
So 0 is attained a either a row or a column. Without loss of generality, we can assume there is row consisting entirely of zeros. (Columns can become rows after a 90 degree rotation about the centre of the grid).

Now 8 can be neither a diagonal sum nor a column sum. So both the 8's are row sums. And then the second 0 is also a row sum.

0	0	0	0	0	0	0	0
1	1	1	0	1	1	1	1
0	0	0	0	0	0	0	0
0	0	0	0	0	0	1	0
1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1
1	1	1	1	1	1	0	1
0	1	0	0	0	0	0	0

The sum 1 occurs twice. Wherever it occurs, there will be one 1 and seven 0's .All the columns , and the two diagonals have at least two ones. So once again, the two sum I's are in the rows.

Look now at sum seven. Wherever this sum occurs, there is exactly one 0. So as before, both sums 7 must occur in the rows. With all eight rows accounted for, the remaining sums must occur in the columns.

Now consider the 4x8 grid who row sums are 1,1,7,7.
(blanks in the diagram). Two of these have exactly one 1, and the other 2 having exactly one zero. **Consider the columns in which these four numbers do not occur.** There are at least four of them and **all have sum 4!** Contradicting the fact that sum 4 occurs exactly twice.

LESSON 2

HCF and the Chinese Remainder Theorem

2.1 HIGHEST COMMON FACTOR

The HCF (highest common factor) of two numbers is firstly, a factor of each of the numbers (that is, **a common factor)**, and secondly, is the largest among those common factors. Take 16 and 24 for example. The common factors are 2, 4 and 8, and the largest among them is 8, so 8 is the HCF of 16 and 24. **We have a way of writing this: we write (16, 24) = 8**

Similarly, (24, 36) = 12, (14, 21) = 7, (25, 36) = 1.

The last example is interesting. 25 and 36 have no common factors other than 1, so 1 is the HCF. There are many such pairs. 18 and 35, 12 and 85, and so on.

We saw earlier that it is not all easy to factorise a given number; we have to divide it by **all the primes that are less than its square root. This may lead us to believe that it is even more difficult to find the HCF of two numbers, since each of these numbers have to factorized. Fortunately , the Chinese found a very clever way of find the HCF, without having to factorise either of the numbers!**

2.2 THE CHINESE REMAINDER THEOREM

The method is the following:

1. Divide the larger (call it a) by the smaller (call it b).
2. The remainder r is smaller than b. (This is always the case, as a little thought will reveal)
3. If it is not 0, repeat, that is, divide the larger by the smaller.
4. Continue until the remainder is 0.
5. **The last remainder that is not 0 is the HCF.**

Let us test this method on the pair 62 and 26. It should be clear that the HCF is 2. Let us use the method above to see whether it works.

62 = 2.26 + 10, 10 < 26
26 = 2.10 + 6, 6 < 10
10 = 1.6 + 4, 4 < 6
6 = 1.4 + 2, 2 < 4
4 = 2.2 + 0
The remainder is 0, the last non-zero remainder is 2, so **(62, 26) = 2**
Let us tackle something more challenging. Determine (1189, 4059).
4059 = 3.1189 + 492, 492 < 1189
1189 = 2.492 + 205, 205 < 492
492 = 2.205 + 82 82 < 205
205 = 2.82 + 41 41 < 82
82 = 2.41 + 0
Hence **(1189, 4059) = 41**

2.3 A PROPERTY OF THE HCF

From the last two examples, we can easily see that

(a) It is possible to write 2, the HCF of 26 and 62, in the form 2 = 26a + 62b, where a and b are **integers**.

(b) It is possible to write 41, the HCF of 1189 and 4059, in the form 41 = 1189a + 4059b, where a and b are **integers**.

Here is how:

(a) 2 = 6 – 1.4 = 6 – 1(10 – 1.6) = 2.6 – 1.10 = 2(26 – 2.10) – 1.10 = 2.26 -5.10 = 2.26 – 5(62 – 2.26) = 12.26 – 5.62 so

2 = 12.26 – 5.62 that is, (a, b) = (12, -5).

(b) 41 = 205 – 2.82 = 205 – 2(492 – 2.205)= 5.205 – 2. 492 = 5(1189 – 2.492) – 2.492
= 5.1189 – 12.492 = 5.1189 – 12 (4059 – 3. 1189) = 41.1189 – 12.4059 so
41 = 41.1189 – 12.4059, that is (a, b) = (41, - 12)

Summarising:

> **2.3.1 Given natural numbers a and b, we can always find integers m and n such that the HCF (a,b) = ma + nb**

2.4 COPRIME NUMBERS

We say that a pair of natural numbers *a* and *b* are **coprime** (or **mutually** prime, or **relatively** prime) if their HCF is 1, that is (*a, b*) = 1. In this case, **none of the prime factors of a can be prime factors of b.**

> **Two numbers are coprime if and only if they have no common prime factors**

Take two numbers that are coprime, like 18 and 35. Ponder over the following question:

Given that 18 is a factor of 35*x*, where *x* is some natural number, what conclusion can you make?

Well, we have 18(…) = 35*x*, that is

2.3.3(…)= 5.7.*x*

This means that if we factorise *x* into a product of prime numbers, each of the primes 2, 3 and 3 on the left will appear in the factorisation of *x* since they are not in 5.7. What this means is **that 2.3.3 = 18 is a factor of *x*.**

Summarising, if 18 is a factor of 35*x* then 18 is a factor of x.

Note that we could make this conclusion purely because **18 and 35 have no common factors**, that is, they are coprime. If we use symbols we have the following **very useful result**:

> **2.4.1 If *c* is a factor of *ab*, and *c* and *a* are coprime, then *c* is a factor of *b*.**

If *p* is a *prime* number, then (*p, a*) is a factor of the prime number *p* and so is either 1 or *p*., and in the latter case, *p* is a factor of *a*. So if a prime number *p* is not a factor of an integer *a,* then *p* and *a* are coprime. Hence we have, from 2.4.1:

> **2.4.3 If a prime number *p* is a factor of *ab*, and *p* is not a factor of *a*, then *p* is a factor of *b.* Which is the same as saying: If a prime *p* is a factor of a product *ab*, then *p* either *p* is a factor of *a* or *p* is a factor of *b.***

2.5 INTEGRAL SOLUTIONS

Often, we are asked to find all **integers (or sometimes, positive integers)** that are solutions to a given equation. The key step usually involves recognizing that *ab = cd* means that *a* is a factor of *cd*, for example. Then 2.4.1 above comes very handy, for if a and c, for example, are coprime, we can conclude that a is a factor of d, and then write down d = ax where x is an integer. This process allows to make further with the problem. **Make sure that you have fully understood the foregoing.**

2.6 EXAMPLES

1. Find all pairs (a; b) of positive integers that satisfy the equation 4a = 5(11 – b).

Solution: Since a is positive, so is 11 – b. The positive integers b for which 11 – b is positive are between 0 and 11.

Now 4 is a factor of 5(11 – b). Since 4 and 5 are coprime, by 2.4.1, we conclude that 4 is a factor of

11 – b. The values of 11 – b, using trhe fact that b is between 0 and 11, are also between 0 and 11. Hence 11 – b is either 4 or 8 and b is either 7 or 3. Substitute these values for b in 4a = 5(11 – b) to get 4a = 20 or 40, that is, a is 5 or 10. Hence there are two solutions for (a ; b) : (5 ; 7) or (10 ; 3)

2. Find all positive integers x, y and z are three positive integers satisfying the equations

$x+y+z=100$...........(1)

$x+5y+\frac{z}{20}=100$......(2)

Solution: We can eliminate x; subtract (1) from (2).

$$4y - \frac{19z}{20} = 0$$

$$80y = 19z$$

Now 80 and 19 are co-prime, and 80 is a factor of 19z. Hence by 2.4.1, 80 is a factor of z. Now z > 0 (given) and < 100, from equation (1).

So z = 80,

80y = 19.80,

y = 19,

x + 19 + 80 = 100, from (1).

x = 1,

There is only one solution: (x, y, z) = (1, 19, 80)

2.7 INTEGRAL SOLUTIONS TO THE EQUATION *ax + by = c*

In this section we try to find **all** pairs (x, y), both numbers being **integers,** that satisfy the linear equation ***ax + by = c***

Examples:

1. **Find all integers x and y such that 2x + 2y = 5**
   Solution: If (a,b) is a solution then 2a + 2b = 5, and a and b are integers. So 2(a + b) = 5. But a + b is an integer. So the left hand side is even, while the right hand side is odd. This is impossible (no number is even and odd at the same time). The **conclusion** is that **no solution exists.**

2. Find all integers x and y such that x + 2y = 5
   Solution; x = 5 – 2y is an integer whenever y is an integer. So the **solution set** is $\{(5-2y, y)\}$ where y is an arbitrary integer. We usually write this as:
   Solution Set = $\{(x, y) = (5-2y, y) \mid y \text{ is an integer}\}$
   or, we can use the symbol $\mathbf{Z}$ where $\mathbf{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ is the set of integers and write:
   Solution Set = $\{(x, y) = (5-2y, y) \mid y \in \mathbf{Z}\}$
   The vertical sign | is "such that". So we read the above as : the set of all (5 – 2y, y) such that y is an integer.
   For example, if y = 200, then x = 5 – 400 = - 395 and (-395) + 2(200) = 5 so (x, y) = (- 395, 200) is a solution. There are infinitely many solutions since we can substitute infinitely many numbers for y.

3. **Determine the solution set of 4x + 11y = 8, where x and y are both integers.**
   Solution: We can rewrite the equation as 11y = 4(2 – x). Since 11 and 4 are coprime we can conclude (**see 2.4.1 above)** that 11 is a factor of 2 – x. So 2 – x = 11n for some integer n. Solving for x, we get
   <u>x = 2 – 11n.</u>

Substituting, 4(2 – 11n) + 11y = 8 . Now solve for y:

11y = 8 – 8 + 44n,

<u>y = 4n</u>.

So (x, y) = (2 – 11n, 4n) where n is any integer. Check:

4(2 – 11n) + 11(4n) = 8 for any value of n so (x, y) = (2 – 11n, 4n) is indeed a solution for every integer n.

Solution Set = $\{(x,y)=(11-2n,4n) \mid n\in\mathbf{Z}\}$

4. Determine the solution set of 67x + 35y = 113, where x and y are both integers.

Solution: Clearly, this is not all obvious. The methods used in the previous examples do not work here. We proceed as follows.

<u>Step 1: Find at least one solution</u>

Solve for y: (or x if you prefer)

$$y=\frac{113-67x}{35}.$$

<u>Step 2 : Divide 113 and -67 by 35 (see Lesson 3: 3.1 & 3.2 on the "division algorithm):</u>

That is, write each in the form 35q + r.

$$y=\frac{113-67x}{35}=\frac{3(35)+8-(35.1+32)x}{35}$$

$$=3-x+\frac{8-32x}{35}$$

<u>Step 3: Find an integer such that $\frac{8-32x}{35}$ is an integer.</u>

For any integer x, the right hand side is an integer **provided 35 is a factor of 8 – 32x = -8(4x - 1).** Since 35 and - 8 are coprime, this happens when 35 is a factor of 4x – 1. Now 35n = 4x – 1 immediately gives a solution for n = 1, namely x = 9. and then

y = $3-x+\frac{8-32x}{35}$ = 3 – 9 + 8(- 1) = - 14.

(x, y) = (9, -14) is a solution of the equation 67x+35y = 113.

Step 3; Let (x, y) be any solution.

So we have:

67x + 35y = 113….(1)

67(9)+35(-14) = 113….(2)

Subtract: 67(x – 9) + 35(y + 14) = 0

67(x – 9) = - 35 (y + 14)

Now 67 and – 35 are coprime. Hence 67 is a factor of y + 14. That is y + 14 = 67n for some integer n.

So y = 67n – 14.

Substitute:

67x + 35(67n – 14) = 113
67x + 67(35n) = 113 + 35.15 = 113 + 490 = 603 = 67.9 so
x + 35n = 9. That is
x = 9 – 35n.

So (x, y) = (9 – 35n, 67n – 14), where n is an arbitrary integer, is a solution. This can be checked by substitution.

Solution set is $\{(x, y) = (9-35n,\, 67n-14) \mid n \in \mathbf{Z}\}$

(You can test this answer. Substitute n = 10 for example and check that (- 341, 656) is indeed a solution of the equation 67x + 35 y = 113).

2.8 EXERCISES

1. Use the Chinese remainder Theorem to find the HCF of 1407 and 4757. (Problem 133)

2. If d is the HCF of 1407 and 4757, determine a solution of the integral equation 1407x + 4757y = d. (Problem 134)

3. Show that 125 and 36 are coprime, and find an integral solution of 125x + 36y = 1. (Problem 135)

4. Prove that the product of the HCF and LCM of two numbers is always equal to the product of the two numbers. (Problem 136)

LESSON 7: PYTHAGORIAN TRIPLES

Two integers have the same **parity** if they are either **both even** or **both odd**.
Clearly

7.1 Two integers have the same parity if, and only if, their sum (or difference) is even.

Consider the identity

$ab + \left(\frac{a-b}{2}\right)^2 = \left(\frac{a+b}{2}\right)^2$ ....(1)

7.2 An integer is a difference of two squares if, and only if, it is a product of two numbers having the same parity.

Proof: If a and b have the same parity, the bracketed numbers in (1) are integers, so one direction is clear. Conversely, let $x = a^2 - b^2$ where a and b are integers. The x is the product of the two integers a – b and a + b, whose sum (2a) is even, so they have the same parity..

For example, 120 = 1.120 = 2.60 = 3.40 = 4.30 = 5.24 = 6.20 = 8.15 = 10.12 gives us all the factorisations of 120. Of these, four pairs of factors have the same parity. Applying (1), we obtain

$120 + 29^2 = 31^2$...................... ½(60 – 2) = 29, ½(60 + 2) = 31
$120 + 13^2 = 17^2$

$120+7^2 = 13^2$
$120 + 1^2 = 11^2$

Now every square number > 4 has a factorization into two numbers having the same parity. This is easy to see; if N is odd,

$N^2 = 1.N^2$ and if N is even, $N^2 = \left(\frac{N^2}{2}\right).2$ gives such a factorization, which lead to the equations:

$$N^2 + \left(\frac{N^2-1}{2}\right)^2 = \left(\frac{N^2+1}{2}\right)^2 \text{...........(if } N \text{ is odd)}$$

$$N^2 + \left(\frac{N^2}{2}-2\right)^2 = \left(\frac{N^2}{2}+2\right)^2 \text{...........(if } N \text{ is even)}$$

A **Pythagorean triple is a triple (x,y,z) such that** $x^2 + y^2 = z^2$

Summarising:

7.3 Every integer $\mathrm{N} \geq 3$ is a lateral side (not the hypotenuse) of a right angled triangle with integral sides.

Conversely, let *N* be a lateral side of a right angled triangle having integral sides.. Then there are positive integers *a* and *b* such that

$N^2 + a^2 = b^2$. Hence,

$N^2 = (b-a)(b+a)$. If we let $x = b - a$ and $y = b + a$ they $x + y = 2b$ is even, so N^2 = *xy* has a factorisation into two numbers of the same parity.

The method we have described produces **all** Pythagorean triples.

So this technique produces all possible right angled triangles having a given side.

Example: Find all right angled triangles having 36 as one of it lateral sides.

Solution: $36^2 = 2.2.2.2.3.3.3.3$ has the following admissible factorisations,

$$2.648 \quad 4.324 \quad 6.216 \quad 8.162 \quad 12.108 \quad 18.72 \quad 24.54$$

There are seven triangles having 36 as one their smaller sides.. From (2), their sides are, respectively,

(36, 323, 325), (36, 160, 164), (36, 77, 85), (36, 48, 60), (36, 27, 45), (36, 15, 39).

7.4 Exercise :

1. Find all the different ways in which the number 2000 can be expressed as the difference of the squares of two positive integers.

2. Find all right angled triangles that contain 12 as a lateral side.

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A **Pythagorean triple is a triple (x,y,z) such that** $x^2 + y^2 = z^2$

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