# LESSON 7: PYTHAGORIAN TRIPLES

Two integers have the same **parity** if they are either **both even** or **both odd**.
Clearly

**7.1 Two integers have the same parity if, and only if, their sum (or difference) is even.**

Consider the identity

$ab + \left(\frac{a-b}{2}\right)^2 = \left(\frac{a+b}{2}\right)^2$................................(1)

**7.2 An integer is a difference of two squares if, and only if, it is a product of two numbers having the same parity.**

**Proof**: If a and b have the same parity, the bracketed numbers in (1) are integers, so one direction is clear. Conversely, let $x = a^2 - b^2$ where a and b are integers. The x is the product of the two integers a – b and a + b, whose sum (2a) is even, so they have the same parity..

For example, 120 = 1.120 = 2.60 = 3.40 = 4.30 = 5.24 = 6.20 = 8.15 = 10.12 gives us all the factorisations of 120. Of these, four pairs of factors have the same parity. Applying (1), we obtain

$120 + 29^2 = 31^2$..................... ½(60 – 2 ) = 29, ½(60 + 2) = 31
$120 + 13^2 = 17^2$

$120 + 7^2 = 13^2$
$120 + 1^2 = 11^2$

Now every square number > 4 has a factorization into two numbers having the same parity. This is easy to see; if N is odd,

$N^2 = 1.N^2$ and if N is even, $N^2 = \left(\frac{N^2}{2}\right).2$ gives such a factorization, which lead to the

equations:

$$N^2+\left(\frac{N^2-1}{2}\right)^2=\left(\frac{N^2+1}{2}\right)^2 \text{...........(if } N \text{ is odd)}$$
$$N^2+\left(\frac{N^2}{2}-2\right)^2=\left(\frac{N^2}{2}+2\right)^2 \text{...........(if } N \text{ is even)}$$

A **Pythagorean triple is a triple (x,y,z) such that** $x^2+y^2=z^2$

Summarising:

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| **7.3 Every integer $\mathrm{N} \geq 3$ is a lateral side (not the hypotenuse) of a right angled triangle with integral sides.** |

Conversely, let *N* be a lateral side of a right angled triangle having integral sides.. Then there are positive integers *a* and *b* such that

$N^2+a^2=b^2$ . Hence,

$N^2=(b-a)(b+a)$. If we let $x=b-a$ and $y=b+a$ they $x+y=2b$ is even, so $N^2$ = *xy* has a factorisation into two numbers of the same parity.

The method we have described produces **all** Pythagorean triples.

So this technique produces all possible right angled triangles having a given side.

**Example**: Find all right angled triangles having 36 as one of it lateral sides.

**Solution**: $36^2=2.2.2.2.3.3.3.3$ has the following admissible factorisations,

$$2.648 \quad 4.324 \quad 6.216 \quad 8.162 \quad 12.108 \quad 18.72 \quad 24.54$$

There are seven triangles having 36 as one their smaller sides.. From (2), their sides are, respectively,

(36, 323, 325), (36, 160, 164), (36, 77, 85), (36, 48, 60), (36, 27, 45), (36, 15, 39).

**7.4 Exercise :**

1. Find all the different ways in which the number 2000 can be expressed as the difference of the squares of two positive integers.

2. Find all right angled triangles that contain 12 as a lateral side.