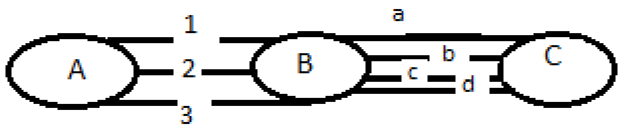
LESSON 4: THE PRODUCT RULE IN COUNTING

4.1 THE PRODUCT RULE IN COUNTING

Suppose A, B and C are three towns. If there are three roads connecting A and B, and four roads connecting B and C, how many different ways are there to get from A to C?



Let us name the roads from A to B by 1, 2 and 3, while the roads from B to C are named a, b, c and d. By 1a, we shall mean the path from A to C, using road 1 followed by road a. If we choose road 1, there are four possible routes from A to C, viz 1a, 1b, 1c and 1d. Likewise, with road 2, we have 2a, 2b, 2c and 2d. And with road three, we have 3a, 3b, 3c and 3d. **Ans: There are 12 ways to get from A to C.**

We could have arrived at 12 using the following argument.

There are* 3 *ways of choosing the first road. For each choice of the first road, there are* 4 *choices for the second road. Hence there are* 3 x 4 = 12 *of choosing both roads together.

We can **<u>generalise</u>**.

> **4.1.1 <u>THE FIRST PRODUCT RULE</u>:** ***Suppose there are m ways of doing thing, and n ways of doing another after the first has been done. Then there are mn ways of doing both things together***.

<u>Examples</u>:

1. How many two lettered "words" can be made using the letters of the word CAT?

<u>Method 1</u>: We can actually make the words.

First letter C: CC, CA, CT

First letter A: AC, AA, AT

First letter T: TC, TA, TT

<u>Answer</u>: 9 words

<u>Method 2 (preferred)</u>: The first letter can be chosen in 3 ways. For each choice of the first letter, there are three choices foe the second letter. So there are 3 x 3 = 9 ways of choosing both letters together, that is, of making two lettered words.

2. How many three lettered words can be made using the letters of the word CAT?

<u>Answer</u>: For each choice of the first two letters, there are three for the third. For example, if the first two were chosen as AT, then we can form ATC, ATA and ATT. But there are 9 ways of choosing the first two letters. So there are 9 x 3 or 3 x 3 x 3 = 27 words that can be made.

The above example gives us a method of solving more complex problems.

3. How many 4 lettered words can be made from the word MASTER?

<u>Answer</u>: The first letter can be chosen in 6 ways, the first two in 6 x 6 ways, the first three in 6 x 6 x 6 ways, so the four letters can be 6 x 6 x 6 x 6 = 1296 ways. So there 1296 such words.

We have the following rule:

<u>4.1.2 THE SECOND PRODUCT RULE IN COUNTING</u> : **Suppose we need to fill in r boxes. If the first box can be filled in m_1 ways, the second in m_2 ways, third in m_3 ways, and so on, until the rth box, which can be filled in m_r ways, then all r boxes can be filled in $m_1 m_2 m_3 .....m_r$ ways.**

m_1	m_2	m_3	....	m_r

4. How many two lettered words can be made from the letters of the word CAT, *if no letter is to be repeated?*

This means that “words”: like CC are <u>not allowed.</u>

The words are: CA, CT, AC, AT, TA, TC.
There are 6 such words.

Or, better still:
The first letter can be chosen in 3 ways, but the second in only two ways, <u>since the first cannot be used.</u>

3	*2*

which leads two 3 x 2 = 6.

5. How many 5 lettered words can be made using the letters from A to G only, and no letter is to be repeated? If repetitions are allowed?

Answer: The letters are to be chosen from the seven letters A, B, C, D, E, F and G. The first letter can be chosen in 7 ways, the second in 6 and so on. So there are

7 x 6 x 5 x 4 x 3 = 2520 words

If repetitions are allowed, we have

7	7	7	7	7

Answer: $7^5 = 16807$ words

4.2 ARRANGEMENTS OR PERMUTATIONS

4.2.1 Definition: Let n be a natural number. For the product 1.2.3.4…..n, we write n! That is n! = 1.2.3…..n. (Read n! as "n factorial").

0! = 1

Examples

1. Show that the number of 5 letter "words", with no letters repeated, that can be made from using the letters of the word STRANGE is $\frac{8!}{3!}$

Proof: We need to fill five boxes with no repetitions:

8	7	6	5	4

So the number of words is 8.7.6.5.4. Now

$$8.7.6.5.4 = 4.5.6.7.8 = \frac{1.2.3.4.5.6.7.8}{1.2.3} = \frac{8!}{3!}.$$

4.3 COUNTING ARRANGEMENTS

Generalise (6) above: Show that the number of r lettered "words" that can made from a word having n different letters, repetitions not allowed, is

$$n(n-1)(n-2)...(n-r+1) = \frac{n!}{(n-r)!}$$

<u>Proof</u>: We have to fill r boxes, without repetitions:

n	n-1	n-2	n-3	….	n-r+1

(The second box is n–1, third is n–2, …so the r^{th} box is n–(r –1) = n –r + 1)

Hence the number of such words is

$$n(n-1)(n-2)...(n-r+1)$$
$$=(n-r+1)(n-r+2)....(n-1)n$$
$$=\frac{1.2.3......(n-r)(n-r+1)....(n-1)n}{1.2.3.....(n-r)}$$
$$=\frac{n!}{(n-r)!}$$

Another way of stating the above result, without resorting to "words" is:

4.2.2 The number of ways of arranging n objects, taken r at a time is

$$n(n-1)(n-2)...(n-r+1)= \frac{n!}{(n-r)!}$$

4.4 **EXERCISES**

1. You are asked to pick the first, second and third athletes in a race in which there are 12 athletes. What is the least number of choices you have to make? If you are given twenty guesses, what is the probability that one of them is correct? (Problem 121)
2. 8 books are to be arranged on a shelf. In how ways can they be arranged? (Problem 122)
3. 8 books are to be arranged on a shelf so that two particular books are to be at the ends. In how many ways can this be done? (Problem 123)
4. How may 5 digit numbers greater than 60000 are odd? (Problem 124)
5. You wish to create a string of 10 numbers using only 0 and 1. How many such numbers can you create? (Problem 125)