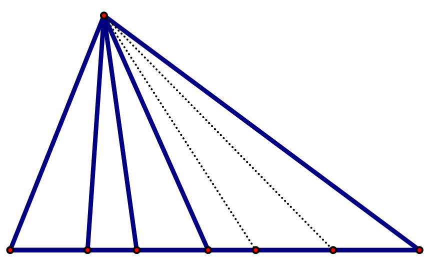
LESSON 5

BINOMIAL COEFFICIENTS

5.1 NUMBERS OF THE FORM $\frac{n(n-1)}{2}$

5.1.1.Reflect on the following problems:

1. Find a formula for $1+2+3+4...(n-1)$ in terms of *n*.
2. Find a formula for $1+2+3+4....+n$
3. Hoe many two-element subsets does a set having n elements have?
4. You have *n* points on a flat surface, no three of which lie on the same straight line. A straight line is drawn through every pair of these points. How many such lines are there?
5. *n* points all lie on the same straight line. A point P that is not on the line is joined to all *n* points. How many triangles are so formed?



6. To win a competition, you have to pick the first two horses in a rece. What is the least number of picks you have to make to be sure of winning, if *n* horses have entered the race?
7. What is the coefficient of x^2 in the expansion of $(1+x)^n$?

<u>Problem 1:</u> Let $x=1+2+3+...+(n-1)$

Write it backwards. The new sum is still x.

$$\begin{aligned} x &= 1 + 2 + ... + (n-2) + (n-1) \\ x &= (n-1) + (n-2) + ... + 2 + 1 \end{aligned}$$

Add both sides:

$$\begin{aligned} 2x &= n+n+n+...+n..........(n-1)\text{ terms} \\ &= n(n-1) \\ x &= \frac{n(n-1)}{2} \end{aligned}$$

Hence

$$\boxed{1+2+3+4...(n-1)=\frac{n(n-1)}{2}}$$

<u>Problem 2</u>: This is the same as Problem 1, except that *n* – 1 is replaced with *n.*

<u>Answer</u>: $1+2+3+4....+n=\frac{(n+1)n}{2}=\frac{n(n+1)}{2}$

5.1.2 The sum of the first n natural numbers is

$$1+2+3+4....+n=\frac{n(n+1)}{2}$$

<u>Problem 3:</u> Let the set be {1, 2, 3, 4, …….*n*}. The two-element subsets are:

{1,2}, {1,3}, {1,4},….{1,n}……………………………there are *n* – 1 of them
{2,3}, {2,4}, {2,5},….{2,n}……………………………. there are *n* – 2 of them
{3,4}, {3,5},{3,6}……{3,n}……………………………. there are *n* – 3 of them
.
.
.
{n,n-1}……………………………………………………. (there is only one)

Note that all the two-element subsets have been counted. And we have

$1+2+3+4...(n-1)$ of them.

<u>So, a set having *n* elements has</u> $1+2+3+4...(n-1)=\frac{n(n-1)}{2}$ **<u>two-element subsets.</u>**

<u>Problem 4</u>: Any two points determine exactly one line, and every such line is associated with exactly one pair of points. <u>So the number of lines is the same as the number of ways two points may be chosen from *n* points.</u> We have seen in problem 3 that this number is $\frac{n(n-1)}{2}$

<u>Problem 5</u>: Three vertices determine a unique triangle; so the number of triangles is equal to the number of ways three points may be chosen. One of them is always P. So we have choose two points from the n points on the line, to create a triangle. The number of ways that this can be done is $\frac{n(n-1)}{2}$

<u>Problem 6 :</u> This is now clear; there are $\frac{n(n-1)}{2}$ ways of picking the two horses.

5.1.3 Let *n* be a natural number. That is $n \in \{1,2,3,4,....\}$. A number of the form $\frac{n(n-1)}{2}$ is called a <u>triangular number</u>. It is also written $\binom{n}{2}$ that is, $\binom{n}{2}$ = $\frac{n(n-1)}{2}$

<u>**Problem 7**</u>: Consider instead the product $(1+x_1)(1+x_2)(1+x_3)....(1+x_n)$. Amongst the terms in its expansion are those which are products of two of the x's, like $x_2x_3,\ x_1x_5,\ x_4x_3$ and so on.How many such terms are there. As many ways as we can choose 2 numbers from 1, 2, 3, …n of course, and thios number is $\binom{n}{2}$ **. Now make all the x's equal to x. That is,** $x = x_1 = x_2 = x_3 ... = x_n$ Then the original product is $(1+x)^n$, all the products of pairs are all equal to x^2,and there are $\binom{n}{2}$ **. of them. So the coefficient of x^2 in the expansion of** $(1+x)^n$ is $\binom{n}{2}$**.**

5.2 <u>Exercises</u>

1. List the first seven triangular numbers. (Problem 149)

2. Consider the following sequence of "triangles" built up from dots:

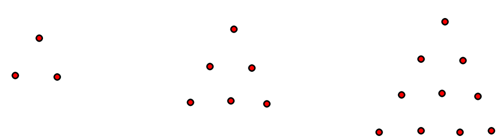


Fig 1 Fig 2 Fig 3

(Problem 150)

1.1 List the number of dots in the first five figures.

1.2 How many dots are there in the 100th figure?

1.3 How many dots are there in Fig. *n*?

1.4 Why are numbers of the type $\frac{n(n-1)}{2}$ called triangular numbers?

1.5 Using ideas from above, can you suggest why numbers of the form n^2 are called square numbers?

5.3 THE BINOMIAL COEFFICIENTS

5.3.1 Reflect on the following problems:

1. A set has five elements. How many subsets, each having 3 elements , does it have?

2. You need to select four vertices of a octagon (an 8-sided polygon). In how many ways can this be done?
3. To win the Lotto, you need to select 6 numbers from 48 numbers. In hour many ways can this done? What are the chances of winning the Lotto?
4. How many subsets does a set of n elements have?

<u>Problem 1</u>:

We list all the 3-element subsets:

{a;b;c}	{a;b;d}	{a;b;e}	{a;c;d}	{a;c;e}	{a;d;e}
{b;c;d}	{b;c;e}	{b;d;e}			
{c;d;e}					

The answer is 10.

Now, if the question required us to count the number of words, each having three letters, that can be made from the letters a, b,c, d and e, we would take the first set {a;b;c} and make the words abc,acb,bac,bca,cab,cba – there are six = 3.2.1 = 3! of them – and do the same for the other 9, thus producing 10 x 6 words altogether. But we saw in the last lesson that the number of three lettered words that can be made from five different letters, no repetitions allowed, was, using the box method,

5	4	3

And so is 5 x 4 x 3.

That is:

The number of three element subsets (that is, 10) times the number of ways three elements can be arranged (that is, 6) = = number of three lettered words that can be made from five letters

Hence 10.6 = 5.4.3 so

$$10 = \frac{5.4.3}{1.2.3}$$

Summarising:

A set having 5 elements has $\frac{5.4.3}{1.2.3}$ **three element subsets.**

The argument above can be used to count the number of subsets having **any number of elements!**

For example:

How many five element subsets does a set having 10 elements have?

Answer:

Let N be the number of such subsets. From the previous example,

The number of five element subsets (x) times the number of ways five elements can be arranged (5!)

= number of five lettered words that can be made from ten letters (10 x 9 x 8 x 7 x 6)

That is,

N x 5! = 10 x 9 x 8 x 7 x 6

$$N = \frac{10.9.8.7.6}{1.2.3.4.5} = 252$$

In general, the problem is to determne the number of r-element subsets a set having n elements has.

We have:

5.3.2 Given a set having n elements, the number of subsets having r elements is $\frac{n(n-1)(n-2).....}{1.2.3.....r}$ **where the numerator and the denominator of the fraction in the formula have the same number (namely, r) of factors.**

Let us go back to Problem 2.

Problem 2:
You need to select four vertices of a octagon (an 8-sides polygon). In how many ways can this be done?

Since the order is not important, the count here is the same as the number of subsets, each having four elements, a ten-element set has.

Answer: $\frac{8.7.6.5}{1.2.3.4} = 70$ ways

Problem 3:
To win the Lotto, you need to select 6 numbers from 48 numbers. In hour many ways can this done? What are the chances of winning the Lotto?

Answer: Again, the number of ways is the same as the number of 6-element subsets a set having 48 elements has, which is

$$\frac{48.47.46.45.44.43}{1.2.3.4.5.6} = 12\ 271\ 512$$

The chances of winning the Lotto are less than 1 in 12 million!

If we multiply the numerator and denominator of the last fraction by 42!, we obtain that the number of 6-element subsets a set of 48 elements has is:

$$\frac{48.47.46.45.44.43.42!}{1.2.3.4.5.6.42!} = \frac{1.2.3....42.43.44...48}{6!42!} = \frac{48!}{6!42!} = \frac{48!}{6!(48-6)!}$$

an answer which uses only 6 and 48.

In general we have

> **5.3.3 Given a set having n elements, the number of subsets having r elements , where r is one of 0, 1, 2, …n, is** $\frac{n(n-1)(n-2).....}{1.2.3.....r} = \frac{n!}{r!(n-r)!}$
>
> **The notation $\binom{n}{r}$ is used to denote the above expression, that is**
>
> $$\binom{n}{r} = \frac{n(n-1)(n-2).....}{1.2.3.....r} = \frac{n!}{r!(n-r)!}$$

The natural number $\frac{n(n-1)(n-2).....}{1.2.3.....r} = \frac{n!}{r!(n-r)!}$ is a **very important number**, so important that it can be found **on hand-held calculators** in the form **nCr** (read as n choose r) or $\binom{n}{r}$.

Try inputting $\binom{10}{5}$, $\binom{8}{4}$ and $\binom{48}{6}$ in nCr to get 252, 70 and 12 271 512, as we have seen.

Numbers of the form $\binom{n}{r}$ are called **binomial coefficients**.

The case r = 0: A subset having 0 elements? Such a set is called the empty set, which is the set having no elements. It is an acceptable set, and is a subset of **every** set .

The following can be easily verified:

$$1. \binom{n}{0} = 1$$

$$2. \binom{n}{r} = \binom{n}{n-r} \text{ for each r = 0, 1, 2, ...n}$$

5.4 THE NUMBER OF SUBSETS OF A SET

Let us now turn to Problem 4

<u>Problem 4</u>

How many subsets does a set of n elements have?

There is a clever way of counting this number. Suppose we want to count how many subsets {1;2;3;4;5;6;7} has. Draw seven boxes as shown:

Take any subset, like {2;4;7}. **Put 1's in columns 2, 4 and 7, and zeros elsewhere.**

0	1	0	1	0	0	1

On the other hand a string of zeros and ones uniquely identifies a subset. Examples are:

1	1	0	0	1	1	1

{1;2;5;6;7}

0	0	0	0	0	0	0

The empty set

0	0	0	0	0	1	0

{6}

So: **To count the number of subsets, we can count instead, the number of strings of zeros and 1's we can create in our box. But this is easy, if we use the product rule! Each box can be filled in only one of two ways:**

2	2	2	2	2	2	2

There are $2^7 = 128$ **subsets altogether.**

Of course, there is nothing special about the number 7. The same argument can be used for a set having n elements, where n is any natural number.

5.4.1 Let a set S have n elements. Then S has 2^n subsets.

5.5 THE EXPANSION OF $(1+x)^n$

A sum having two terms is called a **binomial**. A sum having three terms is called a t**rinomial.** The **binomial theorem** says the following:

$$(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \ldots + \binom{n}{n-1}x^{n-1} + x^n$$

The numbers $\binom{n}{r}$ occur as coefficients of the powers of *x* in the above formula. Hence they are called the binomial coefficients.

5.6 EXERCISES

1. Verify that $\binom{n}{2} = \frac{n(n-1)}{2}$ and $\binom{n}{3} = \frac{n(n-1)(n-2)}{1.2.3}$ (Problem 137)

2. How many 3 element subsets does a set having 8 elements have? (Problem 138)

3. How many 98-element subsets does a set having 100 elements have? (Problem 139)

4. Using a reasonable argument, try to explain why $\binom{100}{98} = \binom{100}{2}$ (Problem 140)
5. A set has 1000 elements. How many subsets does it have? (The Eddington number is an estimate of the number of **electrons in the universe.** Its value is less than 2^{264}, so a set having 1000 elements has more electrons than three universes put together!!)(Bin0mial coefficients) (Problem 141)
6. How many whole numbers in the set 1,2,3….99999 have an odd number of odd digits? (Problem 216)

5.7 PROBABILITY

Lesson 4 and much what we did in this Lesson, answered the question "How many?"
To recall:

- If one thing can be done in m ways, and another , in n ways, after the first has been done, in how many ways can both things be done together? (Answer: mn)
- In how many ways can n objects be arranged? (Answer; n!)
- In how many ways can r objects, taken from n objects, be arranged? (Answer: $n(n-1)(n-2)$.... to r factors)

- In how many ways can r objects be selected from n objects, if order does not matter?

  Answer; $\dfrac{n(n-1)(n-2)....\text{to r factors}}{1.2.3.4...r} = {}^nC_r$

- How many subsets does a set of n elements have? (Answer: 2^n)

To calculate the **probability** that some event is likely to happen, one needs to divide the number of "successes" by the number of "total outcomes" , it is at all possible to do so. **The above formulae are very useful in this regard.**

Example

A coin is tossed 6 times and the outcome, namely H or T (that is, a head falls, or a tail falls), is recorded. Thus HHTHTH is a possible outcome. What is the probability that the record shows three heads?

Solution: Each of the six slots can be filled in two ways, so altogether, there are 2^6 = 64 possible strings. (Second counting principle, Lesson 4.1)

For three H's to occur, we have to select three slots from 6, and this can be done in ${}^6C_3 = \dfrac{6.5.4}{1.2.3} = 20$ ways. Hence the probability is $\dfrac{20}{64} = \dfrac{5}{16}$.