

# LESSON 1

## PRIME NUMBERS AND FACTORISATION

### 1.1 FACTORS:

The natural numbers are the numbers 1, 2, 3, 4, ....

The integers are the natural numbers together with 0 and the negative integers. That is the integers are ...-3, -2, -1, 0, 1, 2, 3,....

This lesson deals mainly with natural numbers and, sometimes, integers. The study of integers is called number theory.

The sum and product of integers (or natural numbers, for that matter), is an integer (natural number).

You will agree that 4 is a **factor** of 8, 6 is a factor of 18, 7 is a factor of 35, and so on. Is -7 a factor of 35? So, what do we really mean by the expression “**a is a factor of b**”?

One way of answering this is:

1.1.1 Let  $a$  and  $b$  be integers, where  $a \neq 0$ . then  $a$  is a factor of  $b$  if  $\frac{b}{a}$  is an integer.

So the answer to the last question is yes, since -5 is an integer. We could say that a factor is a positive integer (which is exactly the same as a natural number), if we so wish.

If we give a name to  $\frac{b}{a}$ , call it  $n$  for example, we can say  $\frac{b}{a} = n$ , where  $n$  is an integer.

Or we can take it further: **a is a factor of b if  $b = an$  where  $n$  is an integer**. While all these can serve as definitions for the statement “ $a$  is a factor of  $b$ ”, the last one is the one that is used most.

For example, the positive factors of 15 are 1, 3, 5 and 15. However the answer to

“Find *all* the factors of 15. “ is : 1, -1, 3, -3, 5, -5, 15, -15.

1. Instead of using the cumbersome “ $a$  is a factor of  $b$ ” we often write  $a|b$  to mean the same thing. Hence  $4|8$ ,  $6|18$ ,  $7|35$  etc.
2. Factors have some very nice and useful properties. Some of them are:
  - 2.1  $1|b$  for any integer  $b$
  - 2.2  $b|b$  for any integer  $b$

- 2.3 If  $a|b$  and  $b|c$ , then  $a|c$
- 2.4 If  $a|b$  and  $a|c$  then  $a|b + c$  and  $a|b - c$
- 2.5 Always,  $a|ab$ .

*Learn these properties by reading them out in sentences and understanding them.*

## 1.2 EXAMPLES

1. Write down all the positive factors of 12. Then write all the factors.

**Solutions**

$12 = 1 \cdot 12 = 2 \cdot 6 = 3 \cdot 4$ . The positive factors are 1, 12, 2, 6, 3, 4.

The factors are 1, -1, 12, -12, 2, -2, 6, -6, 3, -3, 4, -4.

2. Write down all the positive factors of 60.

**Solution:**

$60 = 1 \cdot 60 = 2 \cdot 30 = 3 \cdot 20 = 4 \cdot 15 = 5 \cdot 12 = 6 \cdot 10$

The factors are 1, 60, 2, 30, 3, 20, 4, 15, 5, 12, 6, 10. There are 12 of them.

3. Determine all integers  $n$  that have the property that  $\frac{35}{n-4}$  is an integer.

**Solution:** Notice that the statement “ $\frac{35}{n-4}$  is an integer” is exactly the same as the statement “ $n - 4$  is a factor of 35”! So  $n - 4$  is one of the numbers 1, -1, 5, -5, 7, -7, 35, -35, from which we deduce that  $n$  is one of the numbers 5, 3, 9, -1, 11, -3, 39, -31. There are eight solutions for  $n$ .

## 1.3 PRIME NUMBERS

1.3.1 A **prime number** is a number **greater** than 1, that has exactly two factors, namely, 1 and itself. Or, put in another way, a number that cannot be factorised any further.

One of the most important properties of prime numbers is the following:

1.3.2 Any natural number can be written as a product of prime numbers, and there is only one such factorisation, if we insist that the primes that occur in the factorisation are written in increasing order.

1.3.3 A number which is not a prime number is a **composite number**. Study the examples below carefully. They inform you on how to factorise numbers into products of prime *powers*.

$$120 = 12 \cdot 10 = 3 \cdot 4 \cdot 10 = 3 \cdot 2 \cdot 2 \cdot 2 \cdot 5 = 2^3 \cdot 3 \cdot 5$$

$$3600 = 36 \cdot 100 = 4 \cdot 9 \cdot 10 \cdot 10 = 2 \cdot 2 \cdot 3 \cdot 3 \cdot 2 \cdot 5 \cdot 2 \cdot 5 = 2^4 \cdot 3^2 \cdot 5^2$$

$$3600 = 60 \cdot 60 = 6 \cdot 10 \cdot 6 \cdot 10 = 2 \cdot 3 \cdot 2 \cdot 5 \cdot 2 \cdot 3 \cdot 2 \cdot 5 = 2^4 \cdot 3^2 \cdot 5^2$$

$$1000 = 10^3 = 2^3 \cdot 5^3$$

$$2013 = 3 \cdot 671 = 3 \cdot 11 \cdot 61$$

$$2014 = 2 \cdot 1007 = 2 \cdot 19 \cdot 53$$

$$2015 = 5 \cdot 403 = 5 \cdot 13 \cdot 31$$

1.3.4 Every natural number  $n$  can be written in the form  $2^m d$  where  $d$  is the largest odd factor of  $n$

**Proof:** Let  $n$  be a natural number. Factorise as a product of prime numbers. Then  $n = 2^m d$  ( $m$  may be zero, if  $n$  is odd for example) where  $d$  is a product of prime numbers greater or equal to 3, and hence odd. Clearly  $d$  is the largest odd factor of  $n$ .

## 1.4 NUMBER OF DISTINCT (POSITIVE) FACTORS OF A NUMBER

**Let  $n$  be the number under investigation. How many distinct factors does  $n$  have?**

Firstly suppose  $n$  is prime number. Then it has two factors, namely 1 and  $n$ . Next suppose  $n$  is a power of a prime number. For example, suppose

$n = 3^4 = 81$ . Then its factors are 1, 81, 3, 27, 9. They are all, of course powers of 3, that is,

1,  $3$ ,  $3^2$ ,  $3^3$ ,  $3^4$  which simplify to 1, 3, 9, 27 and 81. So  $3^4$  has 5 factors. If 3 was replaced by another prime number, say 5, we will again obtain 5 factors, namely, 1, 5,  $5^2$ ,  $5^3$  and  $5^4$ . It should be now clear that

**1.4.1 If  $p$  is any prime number then  $p^m$  has  $m + 1$  factors, namely, 1,  $p$ ,  $p^2$ ,  $p^3$ ,  $p^4$ , ...,  $p^m$ .**

Now, any natural number can be written (uniquely) as a product of powers of primes, as we have already seen.

**Example:** For example,  $400 = 4 \times 100 = 16 \times 25 = 2^4 \cdot 5^2$ . How does one count the number of its factors? One way is to go through the factors in pairs:

1, 400, 2, 200, 4, 100, 5, 80, 8, 50, 10, 40, 16, 25, 20  
giving 15 factors altogether.

Well, fix a factor of  $5^2$  say, 5. There are five factors of 400 associated with 5, namely,

1.5, 2.5,  $2^2 \cdot 5$ ,  $2^3 \cdot 5$ ,  $2^4 \cdot 5$

This can be done for **each factor of the three factors of  $5^2$ , namely 1, 5 and 25**. They are

|         |               |                 |                 |                 |
|---------|---------------|-----------------|-----------------|-----------------|
| 1       | 2             | $2^2$           | $2^3$           | $2^4$           |
| 1.5     | 2.5           | $2^2 \cdot 5$   | $2^3 \cdot 5$   | $2^4 \cdot 5$   |
| $1.5^2$ | $2 \cdot 5^2$ | $2^2 \cdot 5^2$ | $2^3 \cdot 5^2$ | $2^4 \cdot 5^2$ |

So again we come to 15 factors. But observe that 15 arises from  $15 = 3 \cdot 5$ . More precisely, **for each of the five factors  $1, 2, 2^2, 2^3$  &  $2^4$  of  $2^4$  three new factors may be obtained obtained, by multiplying each of these five with the three factors of  $5^2$** . Summarising: the number of factors of  $2^4 \cdot 5^2$  is  $(4+1)(2+1) = 5 \cdot 3 = 15$ .

This observation allows one to calculate the number of prime factors of **any natural number** quickly:

1. Factorise the number into prime powers.
2. If  $p^n$  is one of the prime powers in the factorisation, then  $p^n$  has  $n + 1$  factors, as we have seen.
3. Increase each of the powers of the primes obtained by 1 and **multiply** all of them.

**The answer you get is the number of factors of the number that was given.**

Formally stated, we have:

**1.4.2 If  $n = p_1^a p_2^b p_3^c \dots$  is the factorisation of  $n$  into prime powers, then  $n$  has  $(a + 1)(b + 1)(c + 1) \dots$  factors altogether.**

**Examples:**

1. We saw earlier that both 400 has 15 factors and 60 has 12 factors. Note that  $400 = 2^4 \cdot 5^2$  and  $(4+1)(2+1) = 15$   
 $60 = 2 \cdot 3 \cdot 5 = 2^1 \cdot 3^1 \cdot 5^1$  and  $(2 + 1)(1 + 1)(1 + 1) = 12$

2. How many factors does 12 800 have?

Answer:  $12\,800 = 2^9 \cdot 5^2$  has  $(9 + 1) \cdot (2 + 1) = 30$  factors.

## 1.5 DIVISIBILITY RULES

The following rules provide quick and easy ways of deciding whether one of the numbers from 2, 3, 4, 5, 6, 8, 9 or 11 are factors of a given number. We state them now and reserve their proofs for Lesson 3.6

A number is divisible by

- 2 if the last digit is even

- 3 if and only if the sum of its digits is divisible by 3.
- 4 if and only if the last two digits is divisible by 4.
- 5 if the last digit is either 0 or 5.
- 6 if it is divisible by 2 and 3.
- 8 if and only if the last three digits is divisible by 8.
- 9 if and only if the sum of its digits is divisible by 9.
- 11 if and only if the the difference between the “alternate sums” is divisible by 11.

#### EXAMPLES

- 43527684 is divisible by 2 since 4 (last digit) is even
- 56178951 is divisible by 3 since  $5+6+1+7+8+9+5+1 = 42$  is divisible by 3
- 34519068 is divisible by 4 since 68 is.
- 9745875 ends in 5 and so is divisible by 5
- 86594384 is divisible by 8 since  $384 = 8 \cdot 48$  is.
- 675483264 is divisible by 9 since  $6+7+5+4+8+3+2+6+4 = 45$  is divisible by 9
- 358296004 is divisible by 11 since  $(3+8+9+0+4)-(5+2+6+0) = 11$  is divisible by 11.

### 1.6 RATIONAL AND IRRATIONAL NUMBERS

Numbers like  $\frac{1}{2}$ ,  $\frac{-398}{457}$ ,  $\frac{15}{1}$ , ... are called **rational numbers**.

So rational numbers are numbers that can be written in the form  $\frac{m}{n}$ , where **m and n are integers**.

Numbers that **cannot** be written in this form are called **irrational** numbers.

The **positive** rational numbers are called **fractions**.

Consider the numbers that are between 0 and 1. Think of them as being dots located on the number

line. If we divide this line segment into two equal parts, we obtain a dot at the number  $\frac{1}{2}$ . If we divide

it into three equal parts, we get two more dots, at  $\frac{1}{3}$  &  $\frac{2}{3}$ . Into 93 parts we get  $\frac{1}{93}, \frac{2}{93}, \frac{3}{93}, \frac{4}{93}, \dots$ . If we

proceed in this manner, that is, divide the segment in to n parts for all possible natural numbers n, the segment is populated with infinitely many dots, all of which are rational numbers, **and the dots account for all fractions between 0 and 1**. A question now presents itself: are there numbers between 0 and 1, which are **not amongst the dots obtained**? This question is exactly the same as:

*Are **all** the numbers between 0 and 1 rational numbers?*

For a long time, it was held that the number line is made up entirely of rational numbers and nothing else. The bomb fell when through a simple logical argument, it was conclusively proved that the **number**

**whose square is equal to 2 is not a rational number.** Which meant that amongst the points on the number line were these dark objects which could not be obtained by dividing one integer by another, the so called **irrational numbers**.

**1.6.1  $\sqrt{2}$  is an irrational number**

**Proof:** We prove by contradiction. Suppose there is a **rational** number whose square is equal to 2. That is, there are integers  $m$  and  $n$  such that  $\sqrt{2} = \frac{m}{n}$ . *We can assume that  $m$  and  $n$  have no factors in common, that is, it is a simplified fraction.*

$\left(\frac{m}{n}\right)^2 = 2$ . Then  $m^2 = 2n^2$ . So  $m^2$  is even. This forces  $m$  to be even. (Otherwise  $m \cdot m$  would have to be even as well). So **2 is a factor of  $m$** .

Let us write  $m$  as  $m = 2k$  where  $k$  is an integer. Substitute into  $m^2 = 2n^2$  to get  $(2k)^2 = 2n^2$ . Simplifying we get  $n^2 = 2k^2$ . This means that 2 is a factor of  $n^2$ , so, as before, 2 is a factor of  $n$ . Now we have that 2 is a factor of both  $m$  and  $n$ . But we said in the beginning that  $m$  and  $n$  have no common factors! This is the contradiction that we were looking for. Proving that the square root of 2 is an irrational number.

### 1.7 Exercises

Problems 126, 127, 128, 131, 132