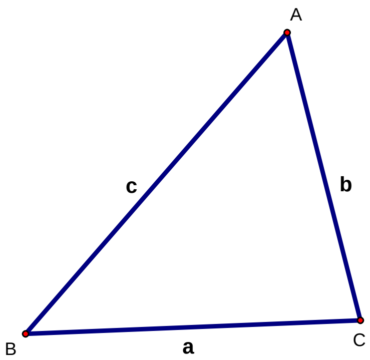
LESSON 11: TRIANGLE FORMULAE

11.1 THE SEMIPERIMETER OF A TRIANGLE



In what follows, Δ ABC will have sides a, b and c, and these will be opposite angles A,B and C respectively. By the triangle inequality,

$a+b>c$

$b+c>a$ and ……………………………….(1)

$c+a>b$

So all of $a+b-c, b+c-a$ & $c+a-b$ are positive real numbers.

The perimeter of Δ ABC is a + b + c and its **<u>semiperimeter</u>** is

$s=\dfrac{a+b+c}{2}$. This quantity plays a very important role in calculations, as we shall presently see. It is easily seen that

$$s-a=\frac{-a+b+c}{2}$$

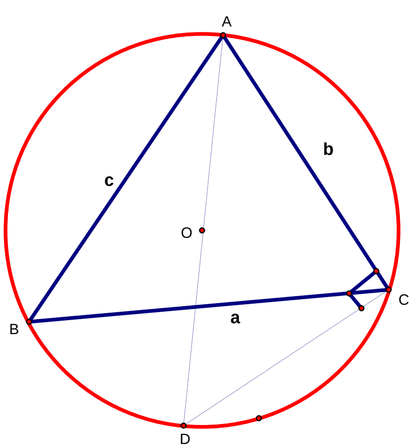
$$s-b=\frac{a-b+c}{2} \quad \text{…………………………(2)}$$

$$s-c=\frac{a+b-c}{2}$$

and by the triangle inequalities, all these are positive.

In this lesson, we shall calculate three important constants associated with a triangle, namely, its **<u>area</u>**, the radius of its inscribed circle (called the **<u>incircle</u>**) and the radius of the (unique) circle that passes through the three vertices, that is, the **<u>circumcircle of</u>** Δ ABC. All these will be calculated in terms of a, b and c. **The symbol R is used to denote radius of the circumscribed circle of Δ ABC.**

11.2 THE SIN RULE



The **sin rule** for a triangle says that the ratio of a side of a triangle to the sin of the angle opposite it, is the same, regardless of the angle selected. More precisely,

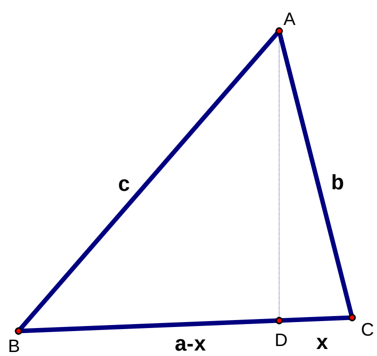
$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

Actually is says a little more, namely, that this constant number is actually **the diameter of the circumscribed circle of Δ ABC.**

Refer to the figure alongside. The circumcentre is O, and the **construction** is to draw the diameter AD through A. So AD = 2R. Now, a diameter of a circle subtends a right angle at the circumference, so angle ACD = 90°. So Δ ACD is a right angled triangle and $\sin D = \frac{b}{2R}$. So $\frac{b}{\sin D} = 2R$. But angles B and D are both subtended by the same chord, namely, AC. Therefore $\angle B = \angle D$ and $\frac{b}{\sin B} = 2R$ = diameter of circumcircle. In like manner, the other two ratios can also be shown to be equal to the diameter. So

THE SIN RULE: $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ **.....................(3)**

11.3 THE COS RULE



There is also a **cosine rule**. It is a rule whereby the cosine of any angle of a triangle can be expressed in terms of the three sides.

The **construction** here is to draw the **altitude AD.** Let x = DC. Then BD = a – x. Using the theorem of Pythagoras, we have the following: $AD^2 = c^2 - (a-x)^2 = b^2 - x^2$

Hence $c^2 = a^2 + b^2 - 2ax$. But $\cos C = \frac{x}{b}$ so $x = b\cos C$, from which follows **the cosine rule:**

$$c^2 = a^2 + b^2 - 2ab\cos C$$

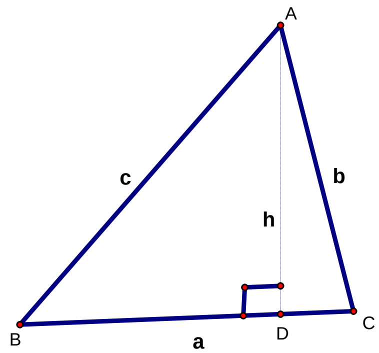
For each side of the triangle, there is a cosine rule:

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = c^2 + a^2 - 2ca\cos B \text{ ....(4)}$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

The sin and the cosine rules hold also for obtuse angled triangles triangles, although our proofs were only for acute angled triangles.

11.4 AREA FORMULA (1):

We are familiar with the formula that gives the area Δ of a triangle as one half the product of its base and height. That is,



$\Delta = \frac{1}{2}ah$ Now $\frac{h}{c} = \sin B$, so $h = c\sin B$. Substitute to get:

$\Delta = \frac{1}{2}ac\sin B$. This gives the area of the triangle, given two sides and an included angle. There is nothing special about angle B; there are three such formula; $\Delta = \frac{1}{2}ac\sin B = \frac{1}{2}ab\sin C = \frac{1}{2}bc\sin A$.

$$\boxed{\Delta = \tfrac{1}{2}ac\sin B = \tfrac{1}{2}ab\sin C = \tfrac{1}{2}bc\sin A, \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5)}$$

11.5 AREA OF A TRIANGLE (2) : HERON'S FORMULA

What is the area of a triangle whose three sides are given? Heron's formula answers this question.

HERON'S FORMULA: The area Δ of a triangle whose sides are a, b and c is given by:

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

where $s = \frac{a+b+c}{2}$ is its semiperimeter.

Proof: From equation 5,

$\Delta = \frac{1}{2}ab\sin C$ so

$$\begin{aligned}
\Delta^2 &= \tfrac{1}{4}a^2b^2\sin^2 C \\
&= \tfrac{1}{4}a^2b^2(1-\cos^2 C) \\
&= \tfrac{1}{4}a^2b^2(1-\cos C)(1+\cos C) \\
&= \tfrac{1}{4}a^2b^2\left[1-\frac{a^2+b^2-c^2}{2ab}\right]\left[1+\frac{a^2+b^2-c^2}{2ab}\right]\ldots\ldots\ldots\ldots\text{(from equation 4)}
\end{aligned}$$

$$
\begin{aligned}
\Delta^2 &= \frac{a^2b^2}{16\mathrm{a}^2b^2}(2ab-a^2-b^2+c^2)(2ab+a^2+b^2-c^2) \\
&= \tfrac{1}{16}\left[c^2-(a^2-2ab+b^2)\right]\left[(a^2+2ab+b^2)-c^2\right] \\
&= \tfrac{1}{16}\left[c^2-(a-b)^2)\right]\left[(a+b)^2-c^2\right] \\
&= \tfrac{1}{16}(c+a-b)(c-a+b)(a+b+c)-(a+b-c) \\
&= \left(\frac{a+b+c}{2}\right)\left(\frac{-a+b+c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{a+b-c}{2}\right) \\
&= s(s-a)(s-b)(s-c)\text{.....................from equation (2)}
\end{aligned}
$$

The result now follows.

11.6 <u>RADII OF THE CIRCUMSCRIBED AND INSCRIBED CIRCLES</u>

11.6.1 The radius of the circumscribed circle

Recall that

$$R=\frac{a}{2\sin A}\text{..........................from equation (3)}$$

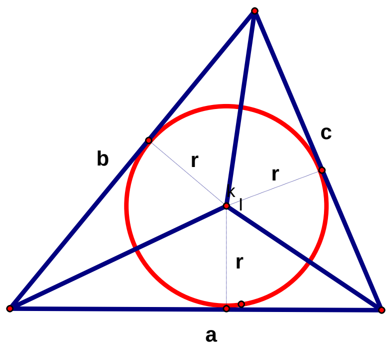
Hence

$$
\begin{aligned}
R &= \frac{a}{2\sin A} \\
&= \frac{abc}{2bc\sin A} \\
&= \frac{abc}{2\Delta}\text{........................from equation (5)}
\end{aligned}
$$

$$\mathrm{R}=\frac{abc}{2\sqrt{s(s-a)(s-b)(s-c)}}\text{.....................................(7)}$$

11.6.2 The radius of the inscribed circle

The three bisectors of the angles of a triangle meet at one point, which is usually named I. If, from I, perpendiculars are drawn to the three sides, **they all have the same length,** say, r. The circle centred at I and having radius r, touches all three sides – the sides are tangent to this circle. It is this r that we shall now calculate.



If Δ denotes the area of the triangle, we see that Δ is the sum of the areas of three triangles, which are swept out when I is joined to the three vertices. If a, b and c are taken to be the bases of these triangles, then they all have the same height, namely, r. We can conclude:

$$\Delta = \tfrac{1}{2}ar + \tfrac{1}{2}br + \tfrac{1}{2}cr = \left(\frac{a+b+c}{2}\right)r = sr$$

Hence

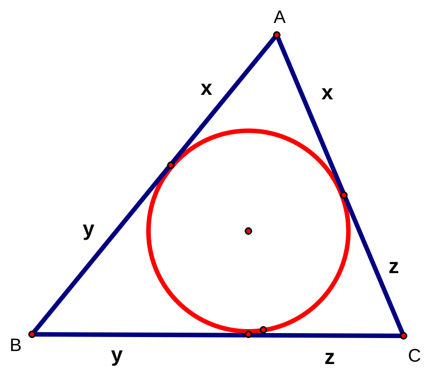
$$\boxed{\begin{aligned} &\Delta = sr \\ &r = \frac{\Delta}{s} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s} \text{..............................}(8) \end{aligned}}$$

using Heron's formula.

11.7 THE TRIANGLE INEQUALITY

Finally, the quantities s – a, s –b, and s – c turn out to be precisely the distance between the vertices and the points of contact with the inscribed circle. More precisely:

Let x, y and z be the lengths of the tangents from the vertices of the triangle, to the points of contact. (See diagram below). The two tangents drawn from a point outside the circle have the same lengths.



From the diagram, the perimeter of the triangle is

2x + 2y + 2z.

But the perimeter is also twice the semiperimeter, that is 2s. Hence 2s = 2x + 2y + 2z, and s = x +y + z. From the diagram again, it is clear that

$$x + y = c$$
$$y + z = a$$
$$z + x = b$$

It therefore follows that

$$
\begin{aligned}
x &= \frac{b+c-a}{2} = s - a \\
y &= \frac{c+a-b}{2} = s - b \\
z &= \frac{a+b-c}{2} = s - c \\
2\Delta &= (x+y+z)\sqrt{xyz} \\
x &> 0, \quad y > 0, \quad z > 0
\end{aligned}
\qquad \text{..}\textbf{(9)}
$$

The last set of inequalities is just another way of writing the triangle inequalities.

So our diagram becomes;