LESSON 8

USEFUL FORMULAE

8.1. **A string of numbers separated by commas is called a <u>sequence</u>.** Examples sequences are:

8.1.1 All natural numbers: 1,2,3,4,…

8.1.2 All squares from 5 onwards: 25, 36, 49, 64,…

8.1.3 All odd cubes from – 3 upwards: - 27, -1, 1, 27, 125,…

The numbers in a sequence are called **terms.** Thus the third term in 8.1.3 is 49, for example.

8.2. How many terms are in the sequence 34, 35, 36, …….237? A neat way of answering the question is to include the numbers 1 to 33 in the sequence and then remove them! So the answer is 237 – 33 = 204.

In general,

8.2 There are n – m + 1 numbers between m and n , both inclusive. That is, the sequence m, m + 1, m + 2, ……,n has n – m + 1 elements

If we include the m - 1 numbers 1, 2, 3, …(m -1) in this sequence and remove them, we deduce that the sequence has n - (m – 1) = n – m + 1 terms.

8.3.

$$\boxed{1+2+3+\ldots n = \frac{n(n+1)}{2}}$$

See lesson 5.1 for an explanation. **This is the formula for the sum of the first *n* natural numbers**. It is also the n^{th} triangular number.

8.4.

The sum of the first n <u>odd</u> numbers is n^2.

Proof: Let us check whether the claim is true for the first few values of n.

1= 1

$1 + 3 = 4 = 2^2$

$1 + 3 + 5 = 9 = 3^2$

$1 + 3 + 5 + 7 = 16 = 4^2$

This test , of course, does not **prove** the statement. It only **verifies** it for the first four values of n**.** How can we be **certain** that if we add the first million odd numbers, for example, we will get million squared, that is, 10^{12}?

So, we have to verify the claim for **every** natural number n.

Method 1:

Let x be the sum. That is, x = 1 + 3 + 5 + …. to n terms. Then

x = (2 – 1) + (4 – 1) + (6 – 1) + ….. n brackets

= (2 + 4 + 6 +….) – (1 + 1 + 1+….) where each bracket has n terms

= 2 (1 + 2 + 3+…) – n

$= 2[\frac{n(n+1)}{2}] - n$ from (2) above

$= n^2 + n - n$

$= n^2$

Method 2:

In the sum x = 1 + 3 + 5 + …., insert the first n even numbers and subtract them:

x = (1 + 2 + 3 + 4 + 5 + 6 +….) – (2 + 4 + 6 +….)

The first bracket is the sum of the first 2n natural numbers and, by problem 2, it is equal to $\frac{(2n)(2n+1)}{2}$. The second bracket is, as before, 2 times the sum of the first n natural numbers, so it is equal to $2.\frac{n(n+1)}{2}$. Substituting:

$$x = \frac{2n(2n+1)}{2} - 2.\frac{n(n+1)}{2}$$

$$= 2n^2 + n - n^2 - n$$

$$= n^2$$

Method 3:

If you check to see how the formula for the sum of the first n natural numbers was derived, (Lesson 5.1.1, problem 2), you will find we resorted to a neat trick. We wrote the sum backwards and added. We try the same trick here. Note first that the n^{th} term (that is, the last term) in

x = 1 + 3 + 5 + ...,

is 2n – 1. The "backward" sum is therefore

x = (2n – 1) + (2n – 3) +.....+ 5 + 3 + 1

So

x + x = 2n + 2n + 2n + ...(n terms)

= n(2n)

Hence $2x = 2n^2$ and $x = n^2$

8.5 Arithmetic Sequences:

The three sequences, namely, the natural numbers, the odd numbers and the even numbers all have a common property.

Each term in the sequence from the second one onwards, can be obtained from the previous term by adding a fixed number.

Another way of saying this is : the difference between consecutive terms (taken in the same "direction" is always the same.

Such a sequence is called an ***arithmetic sequence***.

8.5.1 Let a be the first term of a sequence which has the property that every term is obtained from the previous one by adding a fixed number d. Then the n^{th} term of the sequence is $a+(n-1)d$.

The first term is a, the second term is a + d (adding d to a), the third term is a + 2d (adding d to a +d), the fourth term is a + 3d (adding d to a + 2d). Proceeding in this way, the n^{th} term is $a+(n-1)d$.

8.5.2 The sum of the first n terms of such a sequence is sum $\frac{n}{2}[2a+(n-1)d]$. **that is:**

$$a+(a+d)+(a+2d)+\text{......to } n \text{ terms} = \frac{n}{2}[2a+(n-1)d].$$

Proof: Let $x = a+(a+d)+(a+2d)+\text{......to } n \text{ terms}$

Then $x = (a+a+a+....+a)+(0+1+2+3+...)d$

where each bracket contains n terms. Which means that the second bracket is the sum of the first n – 1 natural numbers.

$$x=(a+a+a+....+a)+(0+1+2+3+...)d$$
$$=na+[(1+2+3+...(n-1)]d$$
$$=na+\frac{(n-1)n}{2}d$$
$$=\frac{n}{2}[2a+(n-1)d].$$

8.5.3 Examples

1. Find

   (a) the 300th term of the arithmetic sequence $-\frac{35}{2},\ -13,\ -\frac{17}{2},\ -4,\ \frac{1}{2},....$

   (b) the value of $-\frac{35}{2}-13-\frac{17}{2}-4+\frac{1}{2}+....$ if there are 200 terms in this sum.

Solution:

(a) The first term of the arithmetic sequence is $a=-\frac{35}{2}$ and the value of d is

$d=-13+\frac{35}{2}=\frac{9}{2}$. (We can check that each of the given terms is obtained by adding $\frac{9}{2}$ of the previous term). The 300th term (so n = 300) is

$$a+(n-1)d=-\frac{35}{2}+(299)(\frac{9}{2})=1328.$$

(b) The terms of the sum are $-\frac{35}{2},-\frac{26}{2},-\frac{17}{2},-\frac{8}{2},\ \frac{1}{2},....$ The first term is $-\frac{35}{2}$ and each term is obtained from the previous one by adding the same number, namely, $\frac{9}{2}$.

So, the above formula may be used, with $a=-\frac{35}{2}$, $d=\frac{9}{2}$ and n = 200. The required sum is

$$\frac{n}{2}[2a+(n-1)d]=\frac{200}{2}[-35+(200-1)\left(\frac{9}{2}\right)]$$
$$=86050$$

8.6 Sums of squares of natural numbers

We have seen that there is a quick way of calculating the sum of the first n natural numbers. Simply use the formula $1+2+3+\ldots n=\frac{n(n+1)}{2}$. What about the squares of natural numbers? Or cubes?

Are there formulae for the sum of the squares, or, for that matter the sums of the cubes of the first n natural numbers?

Let's start with the squares and see whether there is a a pattern :

$$1^2=1$$
$$1^2+2^2=1+4=5$$
$$1^2+2^2+3^2=5+9=14$$
$$1^2+2^2+3^2+4^2=14+16=30$$
$$1^2+2^2+3^2+4^2+5^2=30+25=55$$

At first glance, there does not appear to be any way of predicting what the next sum is going to be: 1, 5, 14, 30, 55, ?? If we divide these numbers by the sum of the associated numbers, we get

$$\frac{1}{1},\frac{5}{3},\frac{14}{6},\frac{30}{10},\frac{55}{15}$$

Which when simplified become

$1.\frac{5}{3},\frac{7}{3},3,\frac{11}{3}$ and this is just the same as $\frac{3}{3}.\frac{5}{3},\frac{7}{3},\frac{9}{3},\frac{11}{3}$, and here there Is a clear pattern!

Summarising, it would appear that;

$$\frac{1^2+2^2+3^2+\ldots+n^2}{1+2+3+\ldots+n}=\frac{2n+1}{3}$$

The denominator on the left is, as we have seen, equal to $\frac{n(n+1)}{2}$ so we suspect that:

$$1^2+2^2+3^2+\ldots+n^2=\left(\frac{n(n+1)}{2}\right)\left(\frac{2n+1}{3}\right)=\frac{n(n+1)(2n+1)}{6}$$

We cannot be certain. We know only that it works for n = 1, 2, 3,4 and 5. So how we can verify that that the formula holds for **every** sum of n squares?

There is a way. A very clever way. We start by making the following observation:

$$(k+1)^3-k^3=k^3+3k^2+3k+1-k^3=3k^2+3k+1$$

$$(k+1)^3-k^3==3k^2+3k+1\ldots\ldots\ldots\ldots\ldots(1)$$

Now take any natural number n.

Make n substitutions for k in the equation (1), namely , k = 1, k = 2, k = 3, ….k = n.

$$2^3 - 1^3 = 3.1^2 + 3.1 + 1$$
$$3^3 - 2^3 = 3.2^2 + 3.2 + 1$$
$$4^3 - 3^3 = 3.3^2 + 3.3 + 1$$
$$5^3 - 4^3 = 3.4^2 + 3.4 + 1.$$

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$$(n+1)^3 - n^3 = 3n^2 + 3n + 1$$

Now add these n equations. All but two terms disappear on the left hand side. On the right, remove 3 as a common factor from the first two columns and *note that the n ones, sum to n.*

$$n^3 + 3n^2 + 3n + 1 - 1 = 3(1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2) + 3.\frac{n(n+1)}{2} + n$$
$$2(n^3 + 3n^2 + 3n) = 6(1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2) + n(3n + 3 + 2)$$
$$6(1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2) = n(2n^2 + 6n + 6 - 3n - 5)$$
$$6(1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2) = n(n+1)(2n+1)$$
$$(1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2) = \frac{n(n+1)(2n+1)}{6}$$

8.6

$$\boxed{1^2 + 2^2 + 3^2 + 4^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}}$$

8.7 Sums of cubes of natural numbers

The sum of the cubes is even more interesting, and surprising. Let's see whether there is a pattern.

$$1^3 = 1$$
$$1^3 + 2^3 = 1 + 8 = 9$$
$$1^3 + 2^3 + 3^3 = 9 + 27 = 36$$
$$1^3 + 2^3 + 3^3 + 4^3 = 36 + 64 = 100$$
$$1^3 + 2^3 + 3^3 + 4^3 + 5^3 = 100 + 125 = 225$$

Here the sums show a clear pattern: $1^2, 3^2, 6^2, 10^2, 15^2$ which is just
$1^2, (1+2)^2, (1+2+3)^2, (+2+3+4)^2, (1+2+3+4+5)^2$

Summarising, we have the fascinating formula

8.7

$$\boxed{1^3 + 2^3 + 3^3 + 4^3 + \ldots n^3 = (1+2+3+4+\ldots+n)^2 = \left[\frac{n(n+1)}{2}\right]^2}$$

As before, we cannot be certain that the formula works for all possible values on n. The proof is identical to the one for squares: start by simplifying $(k+1)^4 - k^4$.

8.8 Geometric sequences

Consider the sequences:
1, 2, 4, 8, 16,……
10, 100, 1000, 10000, …
$\frac{3}{5}, -\frac{2}{5}, \frac{4}{15}, -\frac{8}{45}, \ldots.$
They all have the same property: each term from the second term onwards can be obtained from the previous term by **multiplying** it by a fixed number. The fixed number in the first case is 2, in the second, 10, and in the third $-\frac{2}{3}$. .
Alternately, the **ratio** of consecutive terms (taken in the same direction), is the same throughout the sequence.

Such a sequence is called a ***geometric sequence***

8.8.1 The sum of the first n terms of the geometric sequence $1, r, r^2, r^3, \ldots.$ is $\frac{1-r^n}{1-r}$.

Proof: Let x be equal to the sum. The last term on the right is r^{n-1}. So

$$x = 1 + r + r^2 + r^3 + \ldots. + r^{n-1}$$

Multiply both sides by r and subtract:

$$\begin{aligned}
x &= 1 + r + r^2 + r^3 + \ldots. + r^{n-1} \\
rx &= \quad r + r^2 + r^3 + \ldots. + r^{n-1} + r^n \\
x - rx &= 1 - r^n \\
x(1-r) &= 1 - r^n \\
x &= \frac{1-r^n}{1-r}
\end{aligned}$$

$$1 + r + r^2 + r^3 + \ldots. + r^{n-1} = \frac{1-r^n}{1-r} = \frac{r^n - 1}{r - 1}$$

Multiplying both sides by 1 – r, **we obtain a factorisation of** 1 - r^n

8.8.2

$$\boxed{1 - r^n = (1-r)(1 + r + r^2 + r^3 + \ldots + r^{n-1})}$$

8.9 Other Useful Equations:

The following are stated without proof – they are easy to check – and should be committed to memory.

8.9.1 $x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-xz)$

8.9.2 $(x+y+z)^2=x^2+y^2+z^2+2xy+2yz+2zx$

8.9.3 $(x^2+y^2+z^2-xy-yz-zx)=\frac{1}{2}[(x-y)^2+(y-z)^2+(z-x)^2]$

8.9.4 $a^n-b^n=(a-b)(a^{n-1}+a^{n-2}b+....ab^{n-2}+b^{n-1})$

8.9.5 $(a+b)^n=c_0a^n+c_1a^{n-1}b+c_2a^{n-2}b^2+....+c_nb^n$ where $c_r=\binom{n}{r}=\frac{n!}{(n-r)!r!}$.

8.9.6 $(a+b)^3=a^3+3a^2b+3ab^2+b^3$

8.9.7 $(a+b)^4=a^4+4a^3b+6a^2b^2+4ab^3+b^4$

8.10 **The AG (Arithmetic-geometric inequality)**

The arithmetic mean, that is, the average, of two non-negative numbers a and b can never be less than their geometric mean. That is, for any two numbers a and b,
$\frac{a+b}{2}\geq\sqrt{ab}$ **Moreover, they equal if and only if a = b**

This follows from the fact that

$0\geq(a-b)^2=(a+b)^2-4ab.$ Hence
$\left(\frac{a+b}{2}\right)^2\geq ab$ so $\frac{a+b}{2}\geq\sqrt{ab}$

With equality if and only if $(a-b)^2=0$ that is , a = b.

In general, we have that for any n positive numbers $a_1, a_2, a_3, ...a_n$:

$$\left(\frac{a_1,+a_2+a_3+...a_n}{n}\right)\geq\sqrt{a_1a_2a_3...a_n}$$

with equality if and only if $a_1 = a_2 = a_3 = \ldots a_n$.