The South African Mathematical Olympiad
Third Round 2010
Senior Division (Grades 10 to 12)
Time : 4 hours
(No calculating devices are allowed)

1. For a positive integer n, $S(n)$ denotes the sum of its digits and $U(n)$ its unit digit. Determine all positive integers n with the property that

$$n = S(n) + U(n)^2.$$

Solution: Write n as $a_0 + 10a_1 + 100a_2 + \ldots$, where $a_0, a_1, a_2, \ldots$ are the digits of n. Then the stated equation is equivalent to

$$a_0 + 10a_1 + 100a_2 + \ldots = a_0 + a_1 + a_2 + \ldots + a_0^2$$

or

$$9a_1 + 99a_2 + 999a_3 + \ldots = a_0^2.$$

The right hand side is at most $9^2 = 81 < 99$. Therefore, $a_2, a_3, \ldots$ have to be 0 (otherwise, the left hand side would be strictly greater than the right hand side). Hence we obtain

$$9a_1 = a_0^2.$$

It follows that a_0 must be divisible by 3 (since $9a_1$ is), which leaves us with the possibilities $a_0 = 3$ ($a_1 = 1$), $a_0 = 6$ ($a_1 = 4$) and $a_0 = 9$ ($a_1 = 9$). So there are three solutions: 13, 46 and 99.

2. Consider a triangle ABC with $BC = 3$. Choose a point D on BC such that $BD = 2$. Find the value of

$$AB^2 + 2AC^2 - 3AD^2.$$

Solution: Drop the altitude from A to BC, and let F be its foot. Furthermore, suppose that $BF = x$ (if F lies on the extension of BC beyond B, assign a negative sign to x) and $AF = y$. Then, by the Pythagorean theorem,

$$\begin{aligned} AB^2 &= BF^2 + AF^2 = x^2 + y^2, \\ AC^2 &= CF^2 + AF^2 = (3-x)^2 + y^2, \\ AD^2 &= DF^2 + AF^2 = (2-x)^2 + y^2. \end{aligned}$$

It follows that

$$\begin{aligned} AB^2 + 2AC^2 - 3AD^2 &= x^2 + y^2 + 2(3-x)^2 + 2y^2 - 3(2-x)^2 - 3y^2 \\ &= x^2 + y^2 + 18 - 12x + 2x^2 + 2y^2 - 12 + 12x - 3x^2 - 3y^2 = 6, \end{aligned}$$

regardless of the values of x and y.

3. Determine all positive integers n such that $5^n - 1$ can be written as a product of an even number of consecutive integers.

Solution: $5^n - 1$ cannot be a product of more than five consecutive integers: since one of the factors would have to be divisible by 5, this would have to be the case for the product as well, but $5^n - 1$ is not divisible by 5.

Now we have to consider two cases:

Case 1: If $5^n - 1$ is a product of two consecutive integers, say m and $m+1$, then

$$5^n = m(m+1) + 1 = m^2 + m + 1.$$

Consider this equation modulo 5: clearly, the left hand side is divisible by 5. On the other hand, the right hand side is never divisible by 5:

$m \bmod 5$	0	1	2	3	4
$m^2 + m + 1 \bmod 5$	1	3	2	3	1

It follows that there is no solution in this case.

Case 2: If $5^n - 1$ is a product of four consecutive integers, say $m, \dots, m+3$, then

$$\begin{aligned} 5^n = m(m+1)(m+2)(m+3) + 1 &= (m^2 + 3m)(m^2 + 3m + 2) + 1 \\ &= (m^2 + 3m + 1)^2 - 1 + 1 \\ &= (m^2 + 3m + 1)^2. \end{aligned}$$

Therefore n must be even, and $m^2 + 3m + 1$ must be a power of 5:

$$m^2 + 3m + 1 = 5^k,$$

where $k = \frac{n}{2} > 0$. Solving for m yields

$$m = \frac{-3 \pm \sqrt{5 + 4 \cdot 5^k}}{2} = \frac{-3 \pm \sqrt{5(1 + 4 \cdot 5^{k-1})}}{2}.$$

If now $k > 1$, then $5(1 + 4 \cdot 5^{k-1})$ is divisible by 5, but not by 25 ($4 \cdot 5^{k-1}$ is a multiple of 5 in this case, so $4 \cdot 5^{k-1} + 1$ is not), and so it cannot be a square. This shows that there cannot be any solutions for $k > 1$. On the other hand, $k = 1$ and $m = 1$ satisfy the equation, which leads us to the only solution

$$5^2 - 1 = 1 \cdot 2 \cdot 3 \cdot 4.$$

4. Given n positive real numbers satisfying $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant 0$ and $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$, prove that

$$\frac{x_1}{\sqrt{1}} + \frac{x_2}{\sqrt{2}} + \cdots + \frac{x_n}{\sqrt{n}} \geqslant 1.$$

Solution 1: Note that for any $k \leqslant n$, we have

$$k x_k^2 \leqslant x_1^2 + x_2^2 + \cdots + x_k^2 \leqslant 1$$

by the given conditions. This implies that $x_k \leqslant \frac{1}{\sqrt{k}}$ and thus $x_k^2 \leqslant \frac{x_k}{\sqrt{k}}$. We conclude that

$$\frac{x_1}{\sqrt{1}} + \frac{x_2}{\sqrt{2}} + \cdots + \frac{x_n}{\sqrt{n}} \geqslant x_1^2 + x_2^2 + \cdots + x_n^2 = 1,$$

which proves the statement.

Solution 2: We prove the inequality by induction on n: for $n = 1$, we have $x_1 = 1$ by the given conditions, and there is nothing to prove. Now assume that the inequality holds for $n-1$, and consider real numbers $x_1, x_2, \ldots, x_n$ that satisfy the given conditions. As in the first solution, we find that $x_n \leqslant \frac{1}{\sqrt{n}}$. Now set

$$y_k = \frac{x_k}{\sqrt{1-x_n^2}}.$$

Then $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_{n-1}$ and

$$y_1^2 + y_2^2 + \cdots + y_{n-1}^2 = \frac{1}{1-x_n^2}\left(x_1^2 + x_2^2 + \cdots + x_{n-1}^2\right) = \frac{1}{1-x_n^2}\left(1-x_n^2\right) = 1,$$

so that we can apply the induction hypothesis to $y_1, y_2, \ldots, y_{n-1}$. We conclude that

$$\begin{aligned}\frac{x_1}{\sqrt{1}} + \frac{x_2}{\sqrt{2}} + \cdots + \frac{x_{n-1}}{\sqrt{n-1}} + \frac{x_n}{\sqrt{n}} &= \sqrt{1-x_n^2}\left(\frac{y_1}{\sqrt{1}} + \frac{y_2}{\sqrt{2}} + \cdots + \frac{y_{n-1}}{\sqrt{n-1}}\right) + \frac{x_n}{\sqrt{n}} \\ &\geqslant \sqrt{1-x_n^2} + \frac{x_n}{\sqrt{n}}.\end{aligned}$$

It remains to show that the last expression is $\geqslant 1$ to complete the induction. However, this is equivalent to

$$\sqrt{1-x_n^2} \geqslant 1 - \frac{x_n}{\sqrt{n}}$$

or

$$1 - x_n^2 \geqslant 1 - \frac{2x_n}{\sqrt{n}} + \frac{x_n^2}{n},$$

which finally simplifies to

$$x_n\left(\frac{2}{\sqrt{n}} - \left(1 + \frac{1}{n}\right)x_n\right) \geqslant 0.$$

Since we know that $x_n \leqslant \frac{1}{\sqrt{n}}$, we have

$$\frac{2}{\sqrt{n}} - \left(1 + \frac{1}{n}\right)x_n \geqslant \frac{2}{\sqrt{n}} - 2x_n \geqslant 0,$$

so that both factors on the left hand side are nonnegative. This completes our proof.

5. (a) A set of lines is drawn in the plane in such a way that they create more than 2010 intersections at a particular angle α. Determine the smallest number of lines for which this is possible.

   (b) Determine the smallest number of lines for which it is possible to obtain exactly 2010 such intersections.

Solution: We consider bundles of parallel lines. Each bundle encloses an angle α with at most two other bundles. Each of these bundles possibly encloses an angle α with another bundle, etc. This process comes to an end if either a cycle is closed or the resulting chain of bundles has two ends that enclose an angle α with only one other bundle. If $a_1, a_2, \ldots, a_n$ are the sizes of the bundles involved (the size of a bundle is the number of parallel lines it consists of), then we get

$$a_1a_2 + a_2a_3 + \ldots + a_{n-1}a_n \text{ or } a_1a_2 + a_2a_3 + \ldots + a_{n-1}a_n + a_na_1$$

intersections. The latter is only possible if $n > 2$. First we prove the following: if $a_1 + a_2 + \ldots + a_n = M$ and $n \geq 4$, then

$$a_1a_2 + a_2a_3 + \ldots + a_{n-1}a_n + a_na_1 \leq \frac{M^2}{4}.$$

The proof is by induction on n. For $n = 4$, we have

$$a_1a_2 + a_2a_3 + a_3a_4 + a_4a_1 = (a_1 + a_3)(a_2 + a_4) \leq \frac{(a_1 + a_2 + a_3 + a_4)^2}{4} = \frac{M^2}{4}$$

by the AM-GM inequality. For $n > 5$, consider an index k such that $a_{k-1} + a_{k+1}$ is maximal (indices modulo n). Without loss of generality, let this k be 2. Then we have

$$\begin{aligned}
&a_1a_2 + a_2a_3 + \ldots + a_{n-1}a_n + a_na_1 \\
&\qquad = a_2(a_1 + a_3) + a_n(a_{n-1} + a_1) + a_3a_4 + \ldots + a_{n-2}a_{n-1} \\
&\qquad \leq (a_2 + a_n)(a_1 + a_3) + a_3a_4 + \ldots + a_{n-2}a_{n-1} \\
&\qquad \leq a_1(a_2 + a_n) + (a_2 + a_n)a_3 + a_3a_4 + \ldots + a_{n-2}a_{n-1} + a_{n-1}a_1 \\
&\qquad \leq \frac{(a_1 + a_2 + \ldots + a_{n-1} + a_n)^2}{4} = \frac{M^2}{4}
\end{aligned}$$

by the induction hypothesis. Furthermore, one has

$$a_1a_2 \leq \frac{(a_1 + a_2)^2}{4}$$

and

$$a_1a_2 + a_2a_3 + a_3a_1 \leq \frac{(a_1 + a_2 + a_3)^2}{3},$$

again applying the AM-GM-inequality. Hence a chain or cycle of bundles as described above that comprises M lines in total will yield at most $\frac{M^2}{3}$ intersections. If there are two or more such chains or cycles, the inequality

$$\frac{m_1^2 + m_2^2 + \ldots + m_r^2}{3} \leq \frac{(m_1 + m_2 + \ldots + m_r)^2}{3}$$

shows that the number of intersections at a particular angle α is always at most $\frac{M^2}{3}$, where M is the total number of lines. So the number of lines has to satisfy

$$\frac{M^2}{3} > 2010$$

or $M \geq 78$. Indeed, one can obtain more than 2010 intersections (2028, to be precise), formed by three bundles containing 26 lines each that intersect each other at a $60°$ angle.

For the second part, we try to find a solution similar to the one in (a) that yields exactly 2010 intersections. Considering three bundles intersecting at a $60°$ angle, we have to solve the equation

$$a_1a_2 + a_2a_3 + a_3a_1 = 2010.$$

Note first that at least two of the three parameters a_1, a_2, a_3 must be even for the left hand side to be even. On the other hand, a_1, a_2, a_3 cannot all be even, since the left hand side would be divisible by 4 in this case, which 2010 is not. So two of the three parameters have to be even, the remaining one odd. This shows that the sum $a_1 + a_2 + a_3$ has to be odd, so that we cannot find a solution with $a_1 + a_2 + a_3 = 78$. We therefore aim for a solution with $a_1 + a_2 + a_3 = 79$. Let us first estimate a_1, a_2, a_3. By the AM-GM inequality, we have

$$2010 = a_1a_2 + a_2a_3 + a_3a_1 \le a_1(a_2 + a_3) + \frac{1}{4}(a_2 + a_3)^2 = a_1(79 - a_1) + \frac{1}{4}(79 - a_1)^2$$

and thus

$$0 \ge 3a_1^2 - 158a + 1799.$$

The roots of this polynomial are $\frac{1}{3}(79 \pm \sqrt{844})$, so that

$$16 < \frac{1}{3}(79 - 30) < \frac{1}{3}(79 - \sqrt{844}) \le a_1 \le \frac{1}{3}(79 + \sqrt{844}) < \frac{1}{3}(79 + 30) < 37.$$

The same inequality holds for a_2 and a_3.

Suppose now that a_1 and a_2 are even while a_3 is odd. Then a_1a_2 is divisible by 4, so that we have to have

$$a_3(a_1 + a_2) \equiv 2010 \equiv 2 \bmod 4.$$

This shows that one of a_1, a_2 is divisible by 4 (a_1, say) while the other one is not. Write $a_1 = 4b_1$, $a_2 = 4b_2 + 2$ and $a_3 = 2b_3 + 1$. Then $5 \le b_1 \le 9$ and $4 \le b_2 \le 8$, which only leaves us with a couple of cases to check. If we plug $a_1 = 4b_1$, $a_2 = 4b_2 + 2$ and $a_3 = 79 - a_1 - a_2$ into the equation, we obtain

$$77b_1 + 75b_2 = 4b_1^2 + 4b_1b_2 + 4b_2^2 + 464.$$

Modulo 4, this yields $b_1 \equiv b_2 \mod 4$. So we are left with six cases to check: $(b_1, b_2) \in \{(5,5), (6,6), (7,7), (8,4), (8,8), (9,5)\}$. $(9,5)$ is indeed a solution: we find $a_1 = 36$, $a_2 = 22$ and $a_3 = 21$, which solves the problem with a total of 79 lines.

Let us finally show that it is not possible to solve the problem with 78 lines (by the argument given in (a), less than 78 lines cannot be sufficient). If we have two separate groups of bundles such that bundles from different groups do not intersect at an angle α, then suppose that these two groups contain m_1 and m_2 lines respectively. The argument in (a) now shows that we can obtain at most

$$\frac{m_1^2 + m_2^2}{3} = \frac{(m_1 + m_2)^2 + (m_1 - m_2)^2}{3} \le \frac{78^2 + 76^2}{3} < 2010$$

intersections of the desired type. If we only have one group that forms a chain or cycle as in (a), then we can only get at most

$$\frac{78^2}{4} < 2010$$

intersections, unless there are exactly three bundles that intersect at 60° angles. But this is exactly the case that we treated, which completes our proof.

6. Write either 1 or -1 in each of the cells of a $(2n) \times (2n)$-table, in such a way that there are exactly $2n^2$ entries of each kind. Let the minimum of the absolute values of all row sums and all column sums be M. Determine the largest possible value of M.

   Solution: Split the table into four smaller tables of size $n \times n$. The upper left quarter is now filled with 1s, the lower right quarter with -1s, and each of the remaining two quarters in a checkerboard pattern (if n is odd, fill them in such a way that one of the quarters contains more 1s than -1s, and the other more -1s than 1s). If n is even, then each of the rows and columns contains either $n/2$ 1s and $3n/2$ -1s, or vice versa, so that $M = n$. If n is odd, then each of the rows and columns contains either $(n-1)/2$ 1s (-1s) and $(3n+1)/2$ -1s (1s), or $(n+1)/2$ 1s (-1s) and $(3n-1)/2$ -1s (1s); it follows that $M = n-1$ in this case.

   Now we show that M cannot be larger. If there is a row or column that contains as many 1s as -1s, then $M = 0$, and we are done. Otherwise, split the set of all $4n$ rows and columns into two subsets: those that contain more 1s than -1s, and those that contain more -1s than 1s. One of these two sets must contain at least $2n$ elements. Without loss of generality, assume that there are at least $2n$ rows and columns (of which k are rows and ℓ columns) that contain more 1s than -1s. In each of these rows and columns, there are at least $n + \frac{M}{2}$ 1's and at most $n - \frac{M}{2}$ -1s. The total number of 1s in all these rows and columns is therefore at least

$$(k+\ell)\left(n + \frac{M}{2}\right) - k\ell,$$

   where the last term accounts for those 1s that are possibly double-counted. Hence we have

$$2n^2 \geqslant (k+\ell)\left(n + \frac{M}{2}\right) - k\ell \geqslant (k+\ell)\left(n + \frac{M}{2}\right) - \frac{1}{4}(k+\ell)^2$$

   and thus, with $r = k + \ell$,

$$M \leqslant \frac{2n^2 + r^2/4 - rn}{r/2} = \frac{4n^2}{r} + \frac{r}{2} - 2n = n - \frac{(r-2n)(4n-r)}{2r} \leqslant n,$$

   since $2n \leqslant r \leqslant 4n$ by assumption. Hence we have $M \leqslant n$, which completes the proof in the case that n is even. If n is odd, $M = n$ is impossible, since all row sums and all column sums must be even (sum of an even number of odd numbers), so that we must have $M \leqslant n - 1$.

   We conclude that the largest possible value of M is n if n is even and $n-1$ if n is odd.