South African Mathematical Olympiad
Senior Third Round 2004
Solutions

1. **Solution by Gerhard du Toit (Grade 12, Paarl Boys' High).**

   Let c_n be the number $111\ldots111$ with n ones. Thus, $a = c_{40}$ and $b = c_{12}$. We determine the GCD of a and b using the Euclidean algorithm:

$$\begin{aligned}(a, b) &= (c_{40}, c_{12}) \\ &= (c_{40} - 10^{28}c_{12}, c_{12}) \\ &= (c_{28}, c_{12}) \\ &= (c_{28} - 10^{16}c_{12}, c_{12}) \\ &= (c_{16}, c_{12}) \\ &= (c_{16} - 10^{4}c_{12}, c_{12}) \\ &= (c_4, c_{12}).\end{aligned}$$

   Now, $c_{12} = c_4(10^8 + 10^4 + 1)$, so $(c_4, c_{12}) = c_4 = 1111$.

2. **Solution by Jon Smit (Grade 12, Grey College).**

   By Euler's Theorem, the answer does not depend on the way the polygon is cut up. If V is the number of vertices, E the number of edges and P the number of planar "faces", then Euler's Theorem states that $P + V - E = 2$. Let e be the number of edges that do not form a side of the polygon and let t be the number of triangles. Thus, in our case, $V = 80 + 50 = 130$, $E = e + 80$ and $P = t + 1$ (the "outside" face), giving $e = 49 + t$.

   Now, every edge that does not form a side of the polygon must be the side of exactly two triangles and the 80 sides of the polygon each form side of one triangle. Since each triangle has three sides, we have

$$3t = 2e + 80 = 2(49 + t) + 80 \implies t = 178.$$

   Alternate solution

   If there are t triangles, then the angle sum of all the triangles equals $180°t$. This angle sum is composed of the angles at the vertices of the polygon and the angles around each interior point.

   The interior angles of the polygon add up to $180° \times 80 - 360°$, and the angles around the interior points of course add up to $360° \times 50$. Thus

$$180°t = 180° \times 80 - 360° + 360° \times 50 \implies t = 178.$$

3. Let $f(x) = x\lfloor x\lfloor x\lfloor x\rfloor\rfloor\rfloor$. Clearly, $f(t)$ is a non-decreasing function of t when $t \geqslant 0$ and a non-increasing function of t when $t \leqslant 0$. Since $f(3) = f(-3) = 81$, we look for a value of x which is a little less than -3 or a little greater than 3. If $x < -3$, then

$$\begin{aligned} & \lfloor x\rfloor \leqslant -4 \\ \Longrightarrow\quad & \lfloor x\lfloor x\rfloor\rfloor \geqslant 12 \\ \Longrightarrow\quad & \lfloor x\lfloor x\lfloor x\rfloor\rfloor\rfloor \leqslant -37 \\ \Longrightarrow\quad & x\lfloor x\lfloor x\lfloor x\rfloor\rfloor\rfloor > 111, \end{aligned}$$

showing that there is no solution $x < -3$.

If $x > 3$, then $x = \frac{88}{n}$ where $n = \lfloor x\lfloor x\lfloor x\rfloor\rfloor\rfloor$. Note that n is an integer, and $n \leqslant \frac{88}{3} = 29\frac{1}{3}$. We now test the possible values of n:

$$\begin{array}{llll} n = 29; & \lfloor x\rfloor = 3 & \lfloor x\lfloor x\rfloor\rfloor = \lfloor \frac{264}{29}\rfloor = 9; & n = \lfloor \frac{792}{29}\rfloor = 27 < 29; \\ n = 28; & \lfloor x\rfloor = 3 & \lfloor x\lfloor x\rfloor\rfloor = \lfloor \frac{66}{7}\rfloor = 9; & n = \lfloor \frac{198}{7}\rfloor = 28; \\ n = 27; & \lfloor x\rfloor = 3 & \lfloor x\lfloor x\rfloor\rfloor = \lfloor \frac{264}{27}\rfloor = 9; & n = \lfloor \frac{792}{27}\rfloor = 29 > 27. \end{array}$$

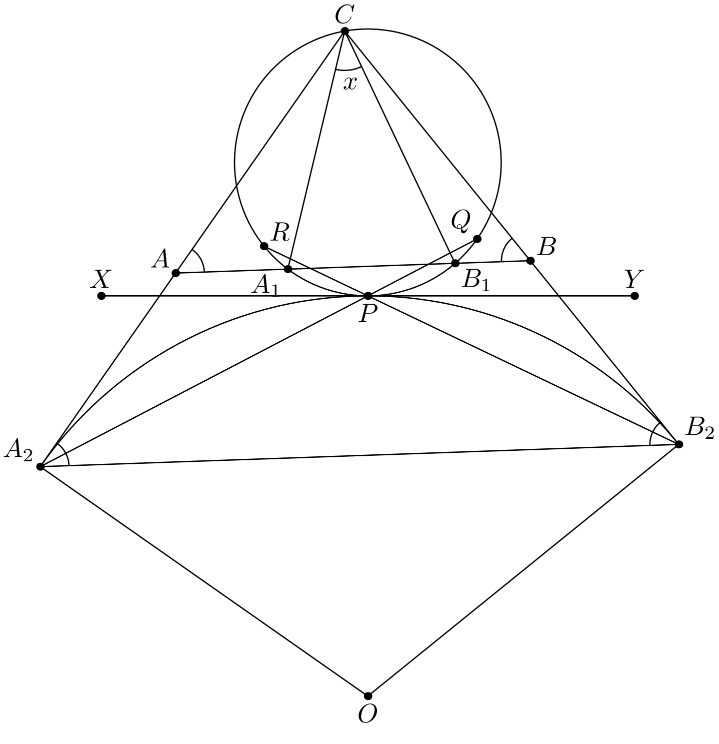
We need not continue any further: since f is non-decreasing, from this point on, there are no solutions. Hence $x = \frac{88}{28} = \frac{22}{7}$.

4. **Solution by Tamara von Glehn (Grade 12, St Stithians Girls' College).**

Since $A_2C = B_2C$ (tangents from the same point) and $AC = BC$, we have that $AB \parallel A_2B_2$. Let the two circles meet in P, let XY be the common tangent at P and let O be the midpoint of the second circle, as shown. Produce B_2P and A_2P to meet the circumcircle of triangle A_1B_1C at R and Q respectively. Let $A_1\widehat{C}B_1 = C\widehat{A}B = C\widehat{B}A = x$.

Since $C\widehat{A_2}O = C\widehat{B_2}O$ are right angles, CA_2OB_2 is cyclic, and so

$$\begin{aligned} A_2\widehat{O}B_2 &= 180^\circ - A_2\widehat{C}B_2 \\ &= 180^\circ - (180^\circ - x - x) \\ &= 2x. \end{aligned}$$



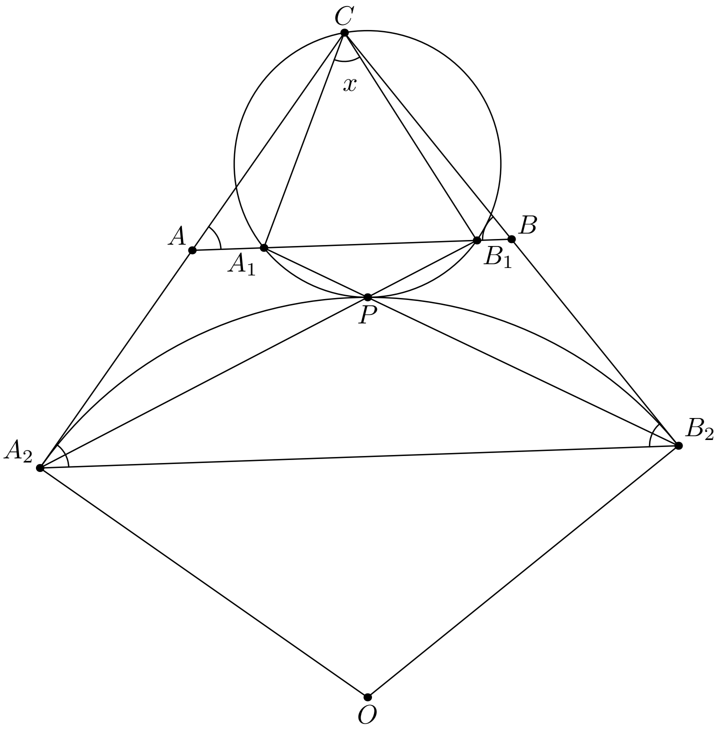
Thus,

$$\begin{aligned} R\widehat{P}Q &= A_2\widehat{P}B_2 \\ &= 180^\circ - \tfrac{1}{2}A_2\widehat{O}B_2 \qquad \text{(Angle at midpoint)} \\ &= 180^\circ - x. \end{aligned}$$

Now, $A_1\widehat{P}B_1 = 180 - x$ since A_1PB_1C is cyclic. Thus $RQ = A_1B_1$, since the two chords subtend equal angles. Also,

$$\begin{aligned} R\widehat{Q}P &= R\widehat{P}X \qquad \text{(Tan-chord)} \\ &= Y\widehat{P}B_2 \qquad \text{(Vertical angles)} \\ &= P\widehat{A_2}B_2, \qquad \text{(Tan-chord)} \end{aligned}$$

so $RQ \parallel A_2B_2$ and thus $RQ \parallel A_1B_1$. Since A_1B_1 and RQ are two equal, parallel chords in the same semi-circle, they are in fact the same chord. Thus A_1PB_2 and B_1PA_2 are straight lines.



In triangles A_2AB_1 and A_1BB_2, we have

$$\begin{aligned} A\widehat{A_2}B_1 &= P\widehat{B_2}A_2 \qquad \text{(Tan-chord)} \\ &= B\widehat{A_1}B_2 \qquad \text{(Alternating angles)} \end{aligned}$$

and similarly, $B\widehat{B_2}A_1 = A\widehat{B_1}A_2$. Thus $\triangle A_2AB_1 \mathbin{|||} \triangle A_1BB_2$ and it follows that $AB_1 \cdot A_1B = AA_2 \cdot BB_2 = AA_2^2$.

In triangles ACB_1 and BA_1C we have $C\widehat{A}B_1 = A_1\widehat{B}C = x$ and

$$\begin{aligned} C\widehat{B_1}A &= C\widehat{B}A + B\widehat{C}B_1 \\ &= x + B\widehat{C}B_1 \\ &= A_1\widehat{C}B, \end{aligned}$$

so $\triangle ACB_1 \ ||| \ \triangle BA_1C$. It follows that $A_1B \cdot AB_1 = CA \cdot CB = CA^2$.

Thus $A_1B \cdot AB_1 = AA_2^2 = AC^2$, implying that $AC = AA_2$. Thus A is the midpoint of CA_2 and $AB \parallel A_2B_2$. By the midpoint theorem, $A_2B_2 = 2AB$, as required.

5. For any n, let A_n be the set of integers $0 \leqslant x < n$ such that x^2 leaves a remainder of 1 when divided by n. Let $f(n) = |A_n|$ be the number of elements in this set. Note that if $k \in A_n$, then $n \mid k^2 - 1 = (k-1)(k+1)$. (The notation $a \mid b$ means b is divisible by a.)

   If n is a power of an odd prime p, then either $p \mid k+1$ or $p \mid k-1$. However, p cannot divide both factors (since their difference is 2) and thus the factor divisible by p must be divisible by n. It follows that $A_n = \{1, n-1\}$ and $f(n) = 2$.

   If n is a power of 2, say $n = 2^m$, it becomes more complicated. For the first few values of m, we find that $f(2) = 1$, $f(4) = 2$, $f(8) = 4$, but the pattern stops here since $f(16) = 4$. In fact, if $m \geqslant 3$, and one of $(k-1)$ and $(k+1)$ is divisible by 2^{m-1}, then the other is divisible by 2 but not by 4. So in general, for n a power of 2 greater than 8, $A_n = \{1, n-1, \frac{n}{2}+1, \frac{n}{2}-1\}$ and $f(n) = 4$.

   Now, let $n = pq$ where p and q are relatively prime. Since p and q are relatively prime, $n \mid k^2 - 1$ if and only if $p \mid k^2 - 1$ and $q \mid k^2 - 1$. Suppose $k \in A_n$. Let $k = a + pr = b + qs$ for $0 \leqslant a < p$, $0 \leqslant b < q$ and some integers r and s. Then p divides $k^2 - 1 = (pr + a)^2 - 1 = (pr)^2 + 2apr + a^2 - 1$, implying that p divides $a^2 - 1$, i.e. $a \in A_p$. Similarly $b \in A_q$. Thus every $k \in A_n$ maps to a pair (a, b) with $a \in A_p$, $b \in A_q$. Conversely, suppose $a \in A_p$ and $b \in A_q$. By the Chinese Remainder Theorem (since p and q are relatively prime), there is a unique number $0 \leqslant k < pq = n$ such that $k = a + pr = b + qs$ for some integers r and s. Now, $k^2 - 1 = (a + pr)^2 - 1 = (pr)^2 + 2apr + (a^2 - 1)$, which is divisible by p. Similarly, $k^2 - 1$ is divisible by q and it follows that $k^2 - 1$ is divisible by n, i.e. $k \in A_n$.

   Thus there is a bijective mapping between the elements $k \in A_n$ and the pairs (a, b) with $a \in A_p$ and $b \in A_q$. It follows that $f(pq) = f(p)f(q)$. We are now ready to calculate $f(n)$ for general n.

   Let $n = 2^m p_1^{a_1} p_2^{a_2} \dots p_t^{a_t}$ have t distinct prime divisors. Then

$$\begin{aligned} f(n) &= f(2^m) f(p_1^{a_1}) \dots f(p_t^{a_t}) \\ &= f(2^m) 2 \cdot 2 \dots 2 \\ &= \begin{cases} 2^{t+2} & \text{if } m \geqslant 3 \\ 2^{t+1} & \text{if } m = 2 \\ 2^t & \text{otherwise.} \end{cases} \end{aligned}$$

6. **Solution by Dirk Basson (Grade 12, Hoërskool Diamantveld).**

Suppose that $a_1,\ a_2$ and a_3 are all prime. Then

$$a_1 \,|\, a_2 + a_3 + a_2a_3 \implies a_1 \,|\, a_1 + a_2 + a_3 + a_1a_2 + a_1a_3 + a_2a_3.$$

Similarly, a_2 and a_3 also divides $a_1 + a_2 + a_3 + a_1a_2 + a_1a_3 + a_2a_3$. Because $a_1,\ a_2$ and a_3 are distinct primes, $a_1a_2a_3$ divides $a_1 + a_2 + a_3 + a_1a_2 + a_1a_3 + a_2a_3$. In other words,

$$\frac{a_1 + a_2 + a_3 + a_1a_2 + a_1a_3 + a_2a_3}{a_1a_2a_3} = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \frac{1}{a_1a_2} + \frac{1}{a_2a_3} + \frac{1}{a_3a_1} \tag{1}$$

is a positive integer.

Since $a_1,\ a_2$ and a_3 are distinct primes, at most one of them can be even, in which case $a_1a_2a_3$ is even, but $a_1 + a_2 + a_3 + a_1a_2 + a_1a_3 + a_2a_3$ is odd and thus cannot be divisible by $a_1a_2a_3$. If all three are odd primes, then the largest possible value of (1) is attained when a_1, a_2 and a_3 are as small as possible, i.e. they take on the values 3, 5 and 7. Thus the maximum value of (1) equals

$$\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{3\cdot 5} + \frac{1}{3\cdot 7} + \frac{1}{5\cdot 7} = \frac{86}{105} < 1,$$

contradicting the fact that it is a positive integer. Thus a_1, a_2 and a_3 cannot all be prime.